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1925

Pennsylvania State College

Mathematics for students of agriculture,

Part Two

THE PENN STATE TRIGONOMETRY

June, 1925.

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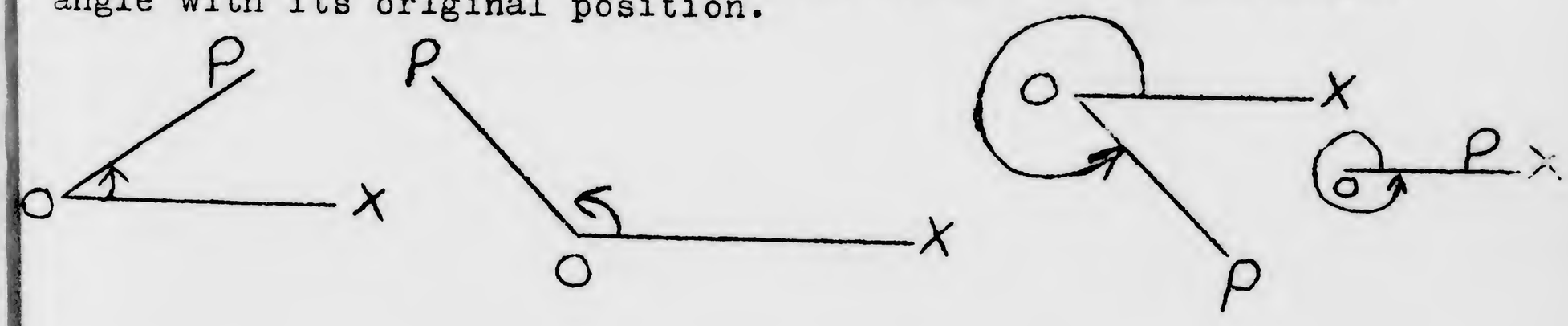
TRIGONOMETRY

The chief object of mathematics is to help us to find unknown values. When we solve a formula or an equation, we find the value of the unknown. Very often in our every day life the unknown values that we need to know are distances between points, or lengths of lines. For example, the length of a table, the heignt of a room, the width of a street, etc.

Direct Measurement. The simplest form of finding distances is that of direct measurement. This consists merely of applying some unit of measure as the foot or the meter to the required distance. This process appears simple enough, but it is very difficult to secure a high degree of precision. There are many sources of error which make our finest measurements only approximations. Name the sources of inaccuracy in measuring a table. How would you measure the diagonal of a room? How would you measure the distance between two points on an uneven piece of ground? Is there any way of knowing whether a measurement is absolutely accurate?

Measurement of Angles. The measurement of angles is scarcely less important than the measurement of lines. An angle is formed by a straight line, as O X rotating in a plane about a fixed point as O until it reaches the position O P. The line O X is called the initial side of the angle. O P is the terminal side. As O P rotates about O, it constantly forms a larger and larger angle with its original position.



Evidently to measure an angle is to measure the amount that the line has rotated about the fixed point.

If the line makes a complete revolution, it turns tnrough a whole circle and reaches its original position. For a long time people have divided the circle into 360 units called degrees and have adopted the degree as the unit of angular measure. The degree (^{o}) is divided into 60 equal parts called minutes ('); and the minute is divided into 60 equal parts called seconds ("). A right angle = 90^o.

Through how many degrees does the minute hand of a clock turn in 15 minutes? in 45' ; in $1\frac{1}{2}$ hours, in 4 hours? Through how many right angles does it turn in each of these periods?

Just as we have a scale for measuring units, we have a scale for measuring angles. The protractor is the instrument used for this purpose. It is an arc of a circle graduated to degrees. The protractor is also used in constructing angles of a required size. For measuring angles in the field surveyers use an instrument called a transit.

1. Use a protractor to construct an angle of 15^{o}, 25^{o}, 72^{o}, 90^{o}.

2. Draw a triangle at random, measure its angles and find their sum. What should this sum be? If you have obtained a different result what are the reasons?

3. Express the following angles in degress: $25^{o}\ 15'$, $10^{o}\ 20'$, $40^{o}\ 45'$, $8^{o}\ 16'\ 12''$.

4. Express the following angles in degrees, minutes, and seconds: $32^{o}.\ 12$; $46^{o}.\ 25$; $15^{o}.\ 215$.

5. Draw a transversal cutting two parallel lines. Measure the corresponding angles. Are they exactly equal? If not, why not?

Indirect Measurements. We have discussed briefly how to measure directly lines and angles. There are many instances, however, in which direct measurement is impossible. For example, consider the case of finding the height of a tree or the distance through a mountain or across a lake or from the earth to the moon. Mathematics makes it possible for us to find such distances indirectly.

Scale drawings. One of the methods used in indirect measurement is that of drawing distances to scale. Scale drawings are familiar in the form of maps, blue prints, architects' plans, and engineers' mechanical drawings. A scale drawing gives an accurate picture of the real object by presenting all the parts in their relative size and position. Obviously this method rests upon the principle of similarity. The ratio between any two parts of the drawing equals the ratio between the corresponding parts of the object. In this manner we are able to determine inaccessible distances and angles in the real object.

In this connection it is well to remember the important facts of similarity proved in geometry:

> If the angles of one triangle are equal respectively to the angles of another triangle, then the triangles are similar and the ratios of the corresponding sides are equal.

By making use of this principle we can find unknown distances by the graphic method of scale drawings.

The plan is as follows:

1. Measure enough lines and angles in the real object so that a scale drawing can be made.

2. Select a unit of distance on the scale to represent a unit of distance in the real object (for example, Scale = 1 into 100 feet).

3. Draw the figure to scale, preferably on squared paper.

4. Measure the lines in the figure which represent the distances in the object which you wish to find.

5. Translate these measurements into the units used in measuring the lines in the real object.

Exercises.

Solve the following graphically by means of the scale and protractor:

1. A boy wishes to find the height of a tree. He measures the shadow and finds it to be 50 feet. At the same time he measures the shadow of a vertical stick 6 feet high and finds it to be $4\frac{1}{2}$ feet long. How can the boy find the height of the tree? What was its height?

2. Place a mirror horizontally on the ground, and stand at a point where the image of the top of a flag pole is just visible in the mirror. Show that by measuring certain distances the height of the pole could be found.

3. Draw an angle of 40°. Make the lengths of its sides 1.5 in. and 2 inches. Join the ends of these sides to complete the triangle. Then draw another angle of 40° and make its sides 3 in. and 4 inches. Complete triangle. Measure the third sides of these triangles. How do they compare? Are the triangles similar?

4. How may the principle found in exercise (3) be used to find the distance across a swamp?

5. In order to find the height of a barn a farmer asked his boy to hold a four-foot stick vertically between the father and the barn so that the lower end of the stick was on a level with the eye of the father. Show how the father found the height of the barn.

6. How may a person on board ship find his distance from the Statue of Liberty, if he knows its height?

7. Find the distance of some object across the street from your home, without crossing the street. Afterward check your result by direct measurement.

8. How may we find the width of a river without crossing the river?

9. Given $a = 2.43$, $b = 3.25$, $A = 55^{\circ}$. Find the remaining parts of the triangle.

10. Given $a = 2.73$, $b = 3.42$, $C = 4.85$. Find the angles. Draw another triangle with its sides twice as large as the given triangle. Measure its angles. How do they compare with the angles of the first triangle?

11. Is a triangle determined when we know the magnitudes of its three angles? Why?

12. A man wishes to find the distance between two trees separated by a barn. How can he find it?

13. A road is inclined $8^{\circ}\ 25'$ to the horizontal. How many feet does it rise per mile, measured along the horizontal?

14. Find the length of a ladder required to reach a window 40 feet above the ground from a point 16 feet in front of the building. What angle would the ladder make with the ground?

15. The length of a kite string is 500 feet and the angle it makes with the horizontal is 50 degrees. Find the height of the kite.

16. A man in a balloon observes that the bases of two towers, which are $1\frac{1}{2}$ apart on a plain subtend an angle of 65 degrees. If he is exactly above the middle point between the towers, find the height of the balloon.

17. Two boys started from the same point. One walked 2 mi. east and 5 miles north, while the other walked 4 miles west and 2 miles south. How far apart were they?

Limitations of the Graphic Method. We have seen how useful and simple the graphic method is in finding indirect measurements. But we can hardly be satisfied with it, for it does not give a high degree of accuracy. Seldom did many of you get the same answer for any problem. A slight error in the drawing means a large error in the object. No one would buy a lot in the heart of a city, if the area of the lot had been determined only by a scale drawing. The main use of the graphic method is to furnish approximate results which serve as checks on results obtained by more accurate methods.

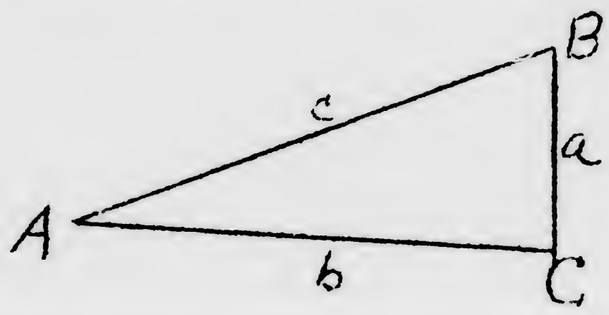
Therefore, we need a more accurate method in measuring lines and angles. Since the measures of lines and angles are usually given as numbers, it seems natural to suppose that there exists a mathematical method to measure unknown distances. This branch of mathematics is known as TRIGONOMETRY, which means triangle-measure.

One Important Object of Trigonometry is to enable us to find the unknown parts of a triangle from the numerical values of the given parts. It forms the basis of surveying, engineering, mechanics and astronomy.

The Right Triangle.

In the last section we learned to find unknown distances by the graphic method. The results obtained, however, were only approximate. We noted that the mathematical method was desirable because of its more accurate results, results as accurate as the direct measurements. The mathematical method consists in the solution of the triangle. We shall first consider a particular kind of triangle which is comparatively easy. It is the Right Triangle.

Consider the right triangle A B C, C being the right triangle. Call the sides small a, b, and c corresponding to the opposite Capitals.



We know from geometry two important relations which help us in the solution of a right triangle, namely:

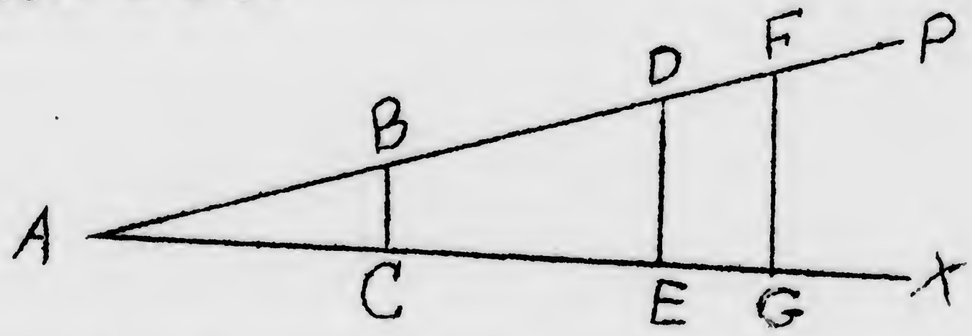
(1) $A + b + C = 180^{\circ}$, or $A + B = 90^{\circ}$

(2) $a^2 + b^2 = c^2$

The first equation enables us to find one of the acute angles if the other is given. The second relation enables us to find any one of the sides if the other two are given.

Note that equation (1) is a relation between the angles alone, while equation (2) is a relation between the sides alone. Consequently, the equation of the angles cannot help us to find the sides, and the equation of the sides cannot help us to find the angles. But this is as far as plane geometry can go towards the solution of a triangle. We need, therefore, certain other relations which unite the sides and angles. Such relations are the foundation of the science of Trigonometry.

In order to understand the nature of these relations, construct an acute angle X A P.



From various points on the line A P drop perpendiculars to A X. The right triangles A C B, A E D, and A G F are all similar (Why?)

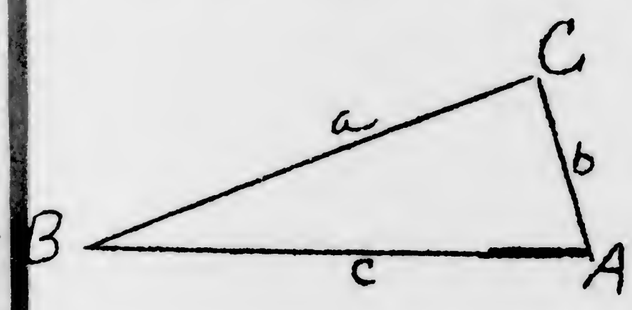
The corresponding sides are, therefore, proportional. That is,

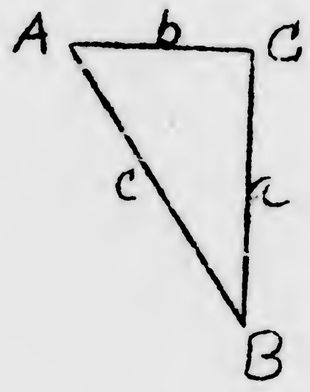
$$\frac{BC}{AC} = \frac{DE}{AE} = \frac{FG}{AG}, \text{ etc.}$$

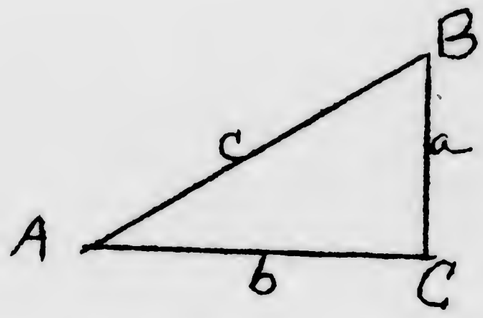
Likewise $\frac{BC}{AB} = \frac{DE}{AD} = \frac{FG}{AF}$, etc.

From this we conclude that the values of these ratios remain the same so long as the angle is unchanged. But if the size of the angle is changed the ratios also change. That is, these ratios depend for their values only upon the size of the angle, and are entirely independent of the lengths of the sides. Hence, the ratios of the sides of a right triangle are functions of an angle. They give another clear example of the function idea. They are called Trigonometric Functions. These functions are the foundation of trigonometry. They unite the sides and angles of a right triangle, and so they enable us to find any part of the triangle, if certain parts are given.

There are altogether six ratios involved in a right triangle, namely, $\frac{a}{c}$, $\frac{b}{c}$, $\frac{a}{b}$, $\frac{b}{a}$, $\frac{c}{b}$, $\frac{c}{a}$.
These ratios are of such importance that each of them has been given a definite name.







In any right triangle,

The ratio of the opposite side to the hypotenuse is called the sine of either acute angle.

The ratio of the adjacent side to the hypotenuse is called the cosine.

The ratio of the opposite side to the adjacent side is the tangent.

The ratio of the adjacent side to the opposite side is thecotangent.

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The ratio of the hypotenuse to the adjacent side is called the <u>secant</u>.

The ratio of the hypotenuse to the opposite side is called the <u>cosecant.</u>

In symbols we may write:

$$\text{Sin } A = \frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\text{Cos } A = \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan A = \frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\cot A = \frac{b}{a} = \frac{\text{adjacent side}}{\text{opposite side}}$$

$$\sec A = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

$$\text{csc } A = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite side}}$$

These definitions should be firmly fixed in mind for they are the foundation of trigonometry. The student should be able to reproduce them without hesitation in whatever position the triangle may be found.

<u>Exercises</u>.

1. Find the functions of 35° by the graphic method.

   <u>Solution</u>. With the protractor construct a right triangle having an angle of 35 degrees. Measure the sides. Find the ratios.

   <u>Note:</u> In finding the sine and cosine make c equal to 1, 10, or 100 to avoid long division. For finding the tangent and secant, make b = 1,10, or 100.

$$\sin 35^\circ = \frac{a}{c} = \frac{5.7}{10} = .57$$

$$\cos 35^\circ = \frac{b}{c} = \frac{8.2}{10} = .82$$

$$\tan 35^\circ = \frac{a}{b} = \frac{7}{10} = .7$$

2. With the scale and protractor find the functions of the following angles:

   12°, 20°, 30°, 40°, 60°, 75°. Construct carefully the following triangles and find the angles and their functions.

3. a = 6, b = 8, c = 10.

4. a = 5, b = 6, c = 10.

5. a = 10, b = 17, c = 18.

6. A floor rises $1\frac{1}{2}$ inches in one foot horizontally. What is its inclination?

7. Given sin A = $\frac{1}{4}$. Construct and measure the angles Find cos A and tan A.

8. Construct and measure an acute angle whose cosine is 2/5. Find its sine and tangent.

9. Construct and measure an angle whose tangent is 5/3. Find its sine and cosine.

10. Show that the sine of an angle cannot be greater than one.

11. Is there an angle whose cosine is 2?

12. Can the cosine of an angle be equal to three?

13. Can the secant of an angle be equal to $\frac{1}{2}$?

14. Are there any restrictions on the values of the tangent?

15. In a right triangle c = 25 inches and sin A = 2/3. Construct the triangle and find the remaining parts.

16. In a right triangle c = 3 feet 4 inches and tan B = 1.2. Construct the triangle and find the remaining parts.

17. Show that in the triangle A C B

sin A = cos B sec A = csc B cos A = sin B

tan A = cot B csc A = sec B cot A = tan B

18. State which of the following is greater and give your reasons:

sin A or tan A cos A or cot A

sec A or tan A csc A or cot A.

19. Draw an angle 30° at the center of a unit circle (a circle whose radius is one unit, one inch, or one foot, one meter). Where one side of the angle cuts the circle draw a tangent prolonging it to meet the other side of the angle. How long is this tangent line? This shows the origin of the name tangent.

20. Construct an isosceles right triangle and find the functions of 45 degrees.

From the preceding exercises we find that

1. Given the value of a trigonometric function, we can find the angle and its other functions.

2. Given an angle, we can construct it and find its functions.

Complementary Angles.

From the definitions of the functions and from the triangle A B C we find

$$\sin A = \frac{a}{c} = \cos B \qquad \cot A = \frac{b}{a} = \tan B$$

$$\cos A = \frac{b}{c} = \sin B \qquad \sec A = \frac{c}{b} = \csc B$$

$$\tan A = \frac{a}{b} = \cot B \qquad \csc A = \frac{c}{a} = \sec B$$

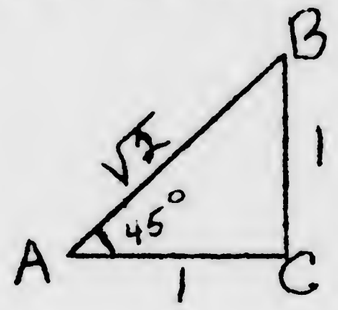
Since A and B are complementary, we may state these results as follows:

Any trigonometric function of the complement of an angle A is equal to the cofuntion of A.

Co-sine means complement's sine; co-tangent means complement's tangent, etc. Thus,, $\sin 70^\circ = \cos(90^\circ - 70^\circ) = \cos 20^\circ$, $\sec 65^\circ\ 20' = \csc(90^\circ - 65^\circ\ 20') = \csc 24^\circ\ 40'$. Hence, if we know the functions of all the angles up to 45°, we know the functions of all angles up to 90°.

Functions of 45°. The functions of certain angles are easily found. Among these are the angles 30°, 45°, and 60°. Let A B C be an isosceles triangle A C and B C being equal to 1.

Then A = B = 45°, and A B = $\sqrt{1+1} = \sqrt{2}$.



Hence,

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2} = .707$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2} = .707$$

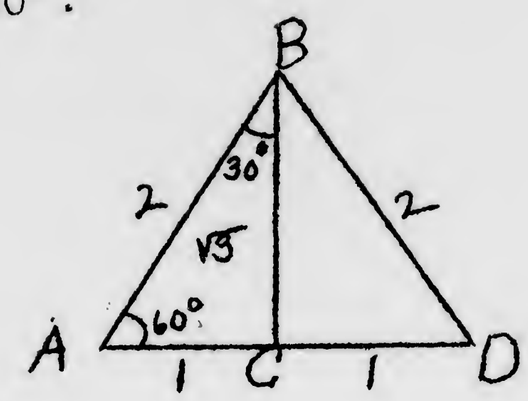
$$\tan 45^\circ = 1$$

$$\cot 45^\circ = 1$$

$$\sec 45^\circ = \sqrt{2} = 1.414.$$

$$\csc 45^\circ = \sqrt{2} = 1.414.$$

Functions of 30o and 60°.



In the equalateral triangle A D B let each side equal 2.
Draw B C perpendicular to A D.
Then by geometry, A C = $\frac{1}{2}$ A B = 1, $\angle$ A B C = $\frac{1}{2}$A = 30°,
and B C = $\sqrt{4 - 1} = \sqrt{3}$.

Note: The hypotenuse is twice the shorter leg. Hence,

$\sin 30^{\circ} = \cos 60^{\circ} = \frac{1}{2}$

$\cos 30^{\circ} = \sin 60^{\circ} = \frac{\sqrt{3}}{2} = .866$

$\tan 30^{\circ} = \cot 60^{\circ} = \frac{1}{\sqrt{3}} = .577$

$\cot 30^{\circ} = \tan 60^{\circ} = \sqrt{3} = 1.732$

$\sec 30^{\circ} = \csc 60^{\circ} = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}.$

$\csc 30^{\circ} = \sec 60^{\circ} = 2.$

The values of the functions of 30°, 45°, and 60° are used very frequently. The student should be able to draw the two triangles instantly and find the functions.

Tables of the Trigonometric Functions. From what has been done it is seen that the values of the functions of any angle may be computed by the graphic method. But it would be too much trouble to do this and the results would not be accurate. Fortunately, mathematicians have calculated these values by the methods of more advanced mathematics and the results have been collected in tabular form known as tables of natural trigonometric functions. With a little practice, the student will soon learn to use these tables readily.

Exercises.

Express as functions of angles less than 45°:

1. $\cos 55^{\circ}$, $\sin 61^{\circ}$, $\tan 75^{\circ}\ 15'$.
2. $\cot 81^{\circ}\ 25'$, $\sec 47^{\circ}\ 40'\ 20''$, $\csc 85.3^{\circ}$.

Write the ratios of the following angles in simplest form:

3. $\sin 30^{\circ}$ to $\tan 60^{\circ}$.

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4. cos 45° to sec 60°.

5. tan 30° to cot 30°.

6. cot 60° to tan 30°.

7. sec 30° to cot 45°.

8. csc 45° to cos 60°.

Find the value of A in the following:

9. $90^\circ - A = \frac{1}{2} A$

10. $45^\circ + A = 60^\circ - 2A$.

11. $\cos A = \sin(45^\circ - \frac{1}{2}A)$. Hint $\sin A = \cos(90^\circ - A)$

12. $\tan 2A = \cot 3A$.

13. $\sec A = \csc (60 - \frac{1}{2}A)$

14. What acute angle is that whose sine is equal to the sine of its complement?

15. Does $\sin 60^\circ = 2 \cdot \sin 30^\circ$? Does $\cos 60^\circ = 2 \cdot \cos 30^\circ$? Is $\sin 2x = 2 \sin x$?

16. Which is greater $\sin 30^\circ$ or $\cos 30^\circ$, $\sin 45^\circ$ or $\cos 45^\circ$, $\sin 60^\circ$ or $\cos 60^\circ$, $\sin 70^\circ$ or $\cos 70^\circ$, $\sin 20^\circ$ or $\cos 20^\circ$?

17. As the angle increases from 0° to 90° how does the value of its sine change? Answer the same question for the other functions.

18. Determine if each of the following equations is true for $x = 30^\circ, 45^\circ, 60^\circ$. $1 + \tan^2 x = \sec^2 x$;

$\cos^2 x + \sin^2 x = 1$; $\tan x = \frac{\sin x}{\cos x}$;

$1 + \cot^2 x = \csc^2 x$.

Are the following relations correct?

19. $\sin 30^\circ \tan 60^\circ = \cos 30^\circ$.

20. $\cos 45^\circ \cdot \sec 45^\circ = 1$.

21. $\cot 60^\circ \cdot \csc 45^\circ = \tan 30^\circ$.

22. $\tan 45^\circ \cdot \sin 60^\circ = \cot 30^\circ$.

Evaluate the following:

23. $\dfrac{\sin^2 60^\circ + \tan 45^\circ - \csc^2 30^\circ}{\cos^2 30^\circ - \cot^2 60^\circ + \sec 60^\circ}$.

24. $10 \cos 30^\circ + 16 \cos 60^\circ - 5\sqrt{3}$

25. $8 \tan 30^\circ - 3 \cot 45^\circ - 8 \cot 60^\circ$.

26. $\sin 30^\circ \cdot \sec 60^\circ \cdot \tan 45^\circ$.

Find the value of A in the following:

27. $\sin A = \dfrac{1}{\sqrt{2}}$, $\cos A = \dfrac{1}{2}$, $\tan A = \sqrt{3}$

$\cot A = 1$, $\sec A = \sqrt{2}$, $\csc A = 2$.

28. $\cos^2 A = \dfrac{1}{2}$, $3\tan A = \sqrt{3}$, $\sin^2 A = \frac{3}{4}$.

The Solution of the Right Triangle.

Any triangle has six elements or parts, namely, the three sides and the three angles, We know from geometry that if any three parts are given, provided one of them is a side, the triangle is determined and may be constructed. By trigonometry the triangle can always be solved when the facts given are sufficient for its construction. To solve a triangle is to find the values of the unknown elements from the given elements.

A triangle may be solved in two ways:

(1) By the graphic method, which is somewhat inaccurate.

(2) By the mathematical method, or Trigonometry.

In this section, we shall consider the solution of the right triangle by the use of trigonometric functions.

To solve a right triangle two elements must be given in addition to the right angle, and one of them must be a side.

There are two possible cases:

(1) When one side and one acute angle are given.

(2) When two sides are given.

Solution Formulas. The formulas usually employed in the solution of a right triangle are:

(1)	$\sin A = \frac{a}{c}$	(6)	$\cos B = \frac{b}{c}$
(2)	$\cos A = \frac{b}{c}$	(7)	$\tan B = \frac{a}{c}$
(3)	$\tan A = \frac{a}{b}$	(8)	$\cot B = \frac{a}{b}$
(4)	$\cot A = \frac{b}{a}$	(9)	$A + B = 90^{o}$
(5)	$\sin B = \frac{b}{c}$	(10)	$c^2 = a^2 + b^2$

Notice that each one of these formulas has three parts. If any two are known, the third may be found. Remember that no formula can be solved which has more than one unknown quantity.

Exercise.

Solve each one of the above ten formulas for every letter in that formula.

Note: (1) If the unknown part is in the numerator, multiply both sides of the formula by the denominator.

(2) If the unknown part is in the denominator multiply both sides of the formula by the denominator. Divide the resulting formula by the multiplier, or coefficient of the unknown.

General Directions for Solving a Right Triangle.

(1) Construct the triangle accurately with scale and protractor. This gives a clear idea of the relation of the parts, helps us to pick the formulas needed, and serves as a check.

(2) Select the formulas to be used and solve them for the unknown parts. Each formula must contain two known parts and the required unknown part. As a rule let the two known parts be the given parts. (Why?)

(3) Make a plan (outline or skeleton) for all the solutions. The arrangement of the work is of great importance. (Why?)

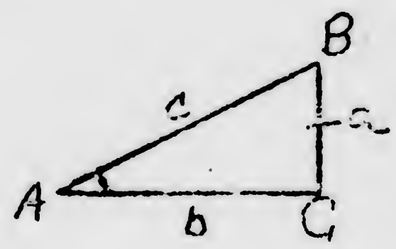
(4) Make your computations carefully by natural functions or by logarithmic functions.

(5) Check your results by the scale drawing and by formulas not used in the solution. The formula
$$a^2 = c^2 - b^2 = (c - b)(c + b)$$
is convenient, and in general it is sufficient. Remember also that the greater side must be opposite the greater angle.

(6) Strive for clearness and neatness.

Example 1.

Given $A = 25^\circ\ 43'$
$a = 16.3$



Find $B = 64^\circ\ 17'$
$b = 33.84$
$c = 37.56$

Formulas: $B = 90^\circ - A = 64^\circ\ 17'$

To find b, $\tan A = \frac{a}{b}$

$b \tan A = a$

$b = \frac{16.3}{.4816}$

$bb = 33.84$

To find c, $\sin A = \frac{a}{c}$

$c \sin A = a$

$c = \frac{a}{\sin A}$

$c = \frac{16.3}{.4339}$

$c = 37.56$

Check: $c^2 = a^2 + b^2$

$c^2 = 1410.75$

$a^2 = 265.69$

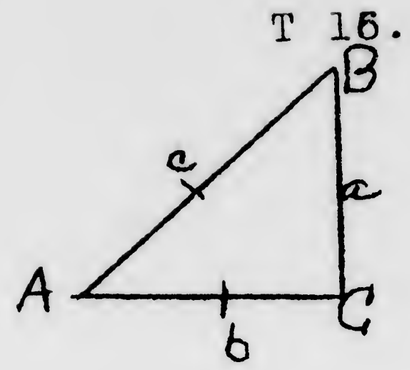
$b^2 = 1145.34$

$a^2 + b^2 = 14\ 11.03$ Checks.

T 16.

Example 2.

Given b = 529.3
c = 902.7



Find A = 54° 53′
B = 35° 7′
a = 731.2[illegible]

Formulas:

To find A, $\cos A = \frac{b}{c}$

$= \frac{529.3}{902.7}$

$= .5752$

$A = 54^\circ\ 53'$

To find a, $\tan A = \frac{a}{b}$

$a = b \tan A$

$= 529.3 \times .3823$

$= 731.24$

$B = 90^\circ - A$

$B = 35^\circ\ 7'$

Check: $a^2 = (c - b)(c + b)$ 373.4 x 1432 = 53470.9

$a^2 = 53471.2$ Checks.

Exercises

Solve the following right triangles using the table of natural functions. Construct them to scale.

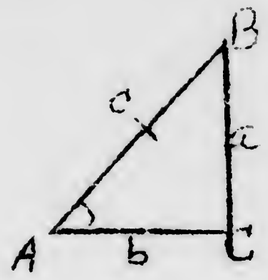
1. A = 33° 40′, a = 57.42
2. A = 27° 15′, b = 16.59
3. B = 40° 14′, a = 23.12
4. B = 52° 10′, c = 73.25
5. b = 0.3712, c = 3.352
6. A = 82° 6′, a = 2.578
7. B = 63° 38′, b = .5732
8. A = 4° 52′, c = 102.35
9. a = .0647, c = .382
10. a = 30.4, b = 25.
11. B = 76° 25′, a = 90.8
12. b = 8, c = 10.5

TIGHT BINDING

The following examples will illustrate the use of the table of the logarithms of functions;

Example 1.

Given $A = 71^\circ\ 13'$
$c = 6.275$



Find $B = 18^\circ\ 47'$
$a = 5.9405$
$b = 2.0205$

From the scale drawing $a = 6$, $b = 2$,

Formulas: $B = 90^\circ - A = 18^\circ\ 47'$

For a, $\sin A = \frac{a}{c}$

$a = c \sin A$
$\log a = \log c + \log \sin A$
$\log 6.275 = 0.79761$
$\log \sin 71^\circ 13' = 9.97623 - 10$
$\log a = 10.77384 - 10$
$a = 5.9407$

For b, $\cos A = \frac{b}{c}$

$b = c \cos A$
$\log b = \log c + \log (\cos A$
$\log 6.275 = 0.79761$
$\log \cos 71^\circ 13' = 9.50784 - 10$
$\log b = 10.30545 - 10$
$b = 2.0205$

Check: $a^2 = (c - b)(c + b)$

$c = 6.275$
$b = 2.0205$
$c - b = 4.2545$
$c + b = 8.2955$

$\log(c - b) = 0.62885$
$\log(c + b) = 0.91884$
$2 \log a = 1.54769$
$\log a = 0.77384$

Since log a is the same as log a obtained in the solution above, the results are correct.

Example 2.

Given $a = .32436$
$b = .06287$

Find $A = 79^\circ\ 1'\ 50''$
$B = 10^\circ 58' 10''$
$c = 0.3304$

Solution be graph gives $A = 11^\circ$, $c = .35$

Formulas:

For A, $\tan A = \frac{a}{b}$

$\log \tan A = \log a - \log b$
$\log 0.32436 = 9.51103 - 10$
$\log 0.06287 = 8.79844 - 10$
$\log \tan A = 0.71259$
$A = 79^\circ\ 1'\ 50''$
$B = 10^\circ 58' 10''$

For c, $\sin A = \frac{a}{c}$

$c \sin A = a$
$c = \frac{a}{\sin A}$
$\log c = \log a - \log \sin A$
$\log 0.32436 = 19.51103 - 20$
$\log \sin 79^\circ 1' 50'' = 9.99200 -$
$\log c = 9.51903 - 10$
$c = .3304$

Check: $a^2 = (c - b)(c + b)$

$c = .33047$
$b = .06287$
$c - b = .26753$
$c + b = .39327$

$\log (c - b) = 9.42737 - 10$
$\log (c + b) = 9.59467 - 10$
$2 \log a = 19.02204 - 20$
$\log a = 9.51102 - 10$

Checks.

TIGHT BINDING

Area. The area of any triangle is equal to one half the product of the base and altitude. Hence, the area of a right triangle is:

$$K = \frac{ab}{2}, \text{ or } 2K = ab.$$

The area may be found by multiplication or by logarithms. If a and b are not given, their logarithms may be found as in the solution of the right triangle.

Exercise.

Construct the following right triangles. Solve them by logarithms and find the areas.

1.	$c = .3281$ $B = 26^\circ\ 43'$	2.	$a = 3.567$ $A = 56^\circ\ 52'\ 40''$	3.	$c = 20.07$ $a = 17.265$
4.	$a = .005$ $b = .00038$	5.	$c = 120.7$ $b = 84.06$	6.	$B = 76^\circ\ 0'\ 15''$ $b = 371.4$
7.	$c = 46.457$ $A = 21^\circ\ 26'\ 32''$	8.	$a = 42.08$ $b = 31.3$	9.	$a = 5479.4$ $B = 83^\circ 50' 4''$
10.	$c = 2090.5$ $b = 708.6$	11.	$B = 78^\circ\ 16\ 33$ $a = .2507$	12.	$a = 7.305$ $b = 10.065$

Isosceles Triangles. The perpendicular from the vertex of an isosceles triangle to the base divides the isosceles triangle into two equal right triangles. Hence any two parts which determine also the isosceles triangle.

Solve the following isosceles triangles in which A and B are the equal angles, a, b, and c are the sides and h is the altitude. Find the areas.

13.	$A = 59^\circ\ 43'$ $b = 405.6$	14.	$B = 34^\circ\ 6'$ $c = .4806$	15.	$c = 74.095$ $b = 105.48$
16.	$c = 4.652$ $C = 72^\circ\ 48'$	17.	$A = 54^\circ\ 16'$ $h = 6.458$	18.	$C = 81^\circ\ 7'\ 20''$ $b = 0.806$

Applications. There is a close connection between trigonometry and surveying. In order to apply trigonometry to surveying, we should learn the meaning of certain terms.

A vertical line is one which coincides with the plumb line.

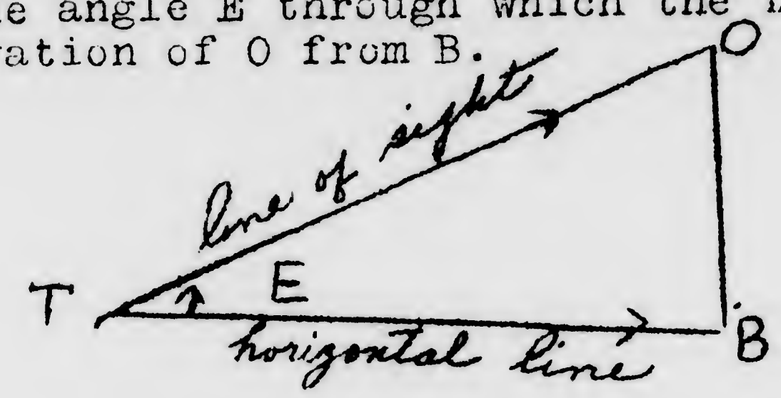
A vertical plane is one which contains a vertical line.

A horizontal plane or line is one which is perpendicular to a vertical line.

The surveyor uses a transit to find angles in the vertical plane and angles in the horizontal plane.

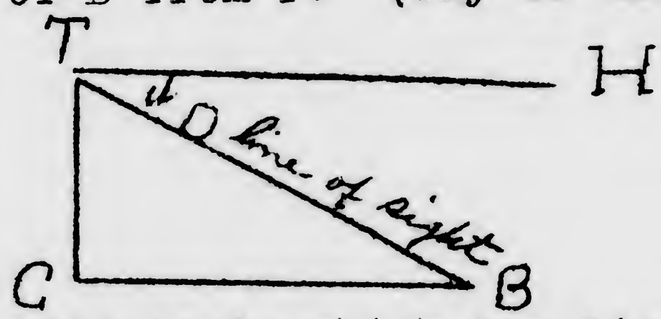
TIGHT BINDING

Angle of Elevation. To find the angle of elevation the transit is placed at T and is first pointed horizontally toward the steeple B O whose height is being measured. The farther end of the telescope is then raised (elevated) until O is in the line of sight. The angle E through which the telescope turns is the angle of elevation of O from B.



E = angle of elevation.

Angle of Depression. If the transit on the top of a tower T is first pointed horizontally (zero reading) and then the farther end is lowered (depressed) until the telescope points to an object at B, the angle through which the telescope turns is the angle of depression of B from T. (Why is the angle H T B = angle T B C?)



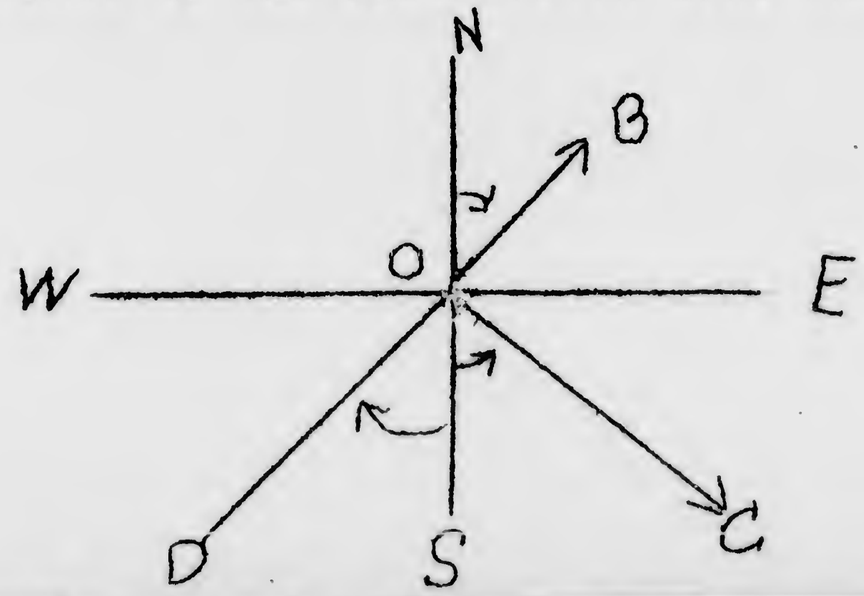
That is to say, the angle which the line of sight from the observer to an object makes with a horizontal line (from the observer), in the same vertical plane, is called the angle of elevation if the object is above the observer, and the angle of depression if the object is below the horizontal plane of the observer.

The angle subtended by a line is that which is obtained by joining the extremities of the line to the eye of the observer. The vertex is at the eye.

Angles in the Horizontal Plane. Bearing. The direction or bearing of a line is the acute angle which the line makes with the north-south line (or meridian). Thus the bearing of OB in the figure is 25 east of north; the bearing of OC is 75 east of south, OD is 50 west of south. These are written more briefly:

N. 25 E, S. 75 E., and S 50 W.

The bearing of a point P from a point O is the bearing of the line OP with reference to the North-South line through O.



TIGHT BINDING

Problems.

1. From a ship the angle of elevation of the top of a lighthouse 250 ft. above the water is $4^\circ 20'$. Find the distance.

2. Find the height of a mountain distant 12560 ft., angle of solution $37^\circ 25'$.

3. In a right triangle c = 127.8 and b = 98.06, find the perpendicular from C to the hypotenuse.

4. The angle of elevation of a road is $10^\circ 12'$. If a man walks a mile up the road, how many feet has he risen?

5. How far would a swimmer be from the Statue of Liberty, if its top (301 ft. above the water) had an elevation angle of $17^\circ 30'$?

6. A ship is sailing 25° north of east at the rate of 15.2 miles per hour. How fast is it going northward and how far eastward?

7. The length of a kite string is 375 yards and the angle of elevation of the kite is $51^\circ 20'$. Find the height of the kite (supposing the string to be straight).

8. A vertical pole 12 feet long casts a horizontal shadow of 9.341 feet. Find the angle of elevation of the sum.

9. The angle of elevation of the sum is $71^\circ 30'$ and the length of the shadow of a tree is 53 feet. 3 inches. Find the height of the tree.

10. The shadow of a steeple is 50.4 feet longer when the altitude of the sun is 32° than when it is $49^\circ 30'$. Find the height of the steeple.

11. A man in a balloon observes that two towers one mile apart subtend an angle of $64^\circ 40'$. If he is exactly above the middle point between the towers, how high is he?

12. A ladder 44 feet long leans against a wall and reaches to a point 32 feet from the ground. Find the distance of the foot of the ladder from the house and the angle between the ladder and the house.

13. Find the length of the side of an equilateral triangle inscribed in a circle whose radius is 7.35 feet.

14. Find the length of the side of a regular pentagon inscribed in a circle wnose radius is 10.5 feet.

15. Find the length of a chord subtending a central angle of $53^\circ 23' 30''$ in a circle whose radius is 135.4 feet.

16. A circular bridge 9.6 feet high has a span 12 feet wide. Find the radius of the circular bridge.

17. A mountain road is inclined $5^\circ\ 15'\ 25''$ to the horizontal. How many feet does it rise per horizontal mile?

18. Find the altitude and area of a parallelogram whose sides are 12 feet and 22 feet and the angle between them is $72^\circ\ 4'$.

19. A man surveying a mine measures a length AB = 340 feet due west with a dip of $7^\circ 20'$; then a length BC = 207 feet due south with a dip of $11^\circ 15'$. How much deeper is C than A?

20. A barn 64 feet long and 48 feet wide has a roof inclined at 42°. Find the area of the roof and the length of the rafter when the roof has a ridge. Find them when the roof has no ridge. How do the results compare?

21. A cylindrical tank, whose axis is horizontal, is 7.2 feet in diameter and 12 feet long. The tank is partly filled with water. The greatest depth of the water is 4.5 feet. How many gallons of water are in the tank?

22. The Washington Monument is 555 feet high. How far apart are two observers who from points due east of the base of the monument observe its angles of elevation to be 25° and $42^\circ\ 20'$ respectively?

23. A man standing on the bank of a river observes that the angle of elevation of the top of a tree on the opposite bank is $61^\circ\ 35'$; when he retires 105 feet from the edge of the river the angle of elevation is 40°. Find the height of the tree and the width of the river.

24. From the top of a tower 140 feet high the angles of depression of the top and bottom of a chimney are found to be 34° and 45° respectively. Find the height of the chimney and its distance from the tower.

25. In the side of a hill that slopes upward at an angle of 28°, a tunnel is bored sloping downward at an angle of $10^\circ\ 35'$ with the horizontal. How far is a point 90 feet down the tunnel below the surface of the hill?

26. The average slope of the bed of a river is $5^\circ\ 5'$ with the horizon. How far does it descend in one mile of its course?

27. From the top of a hill the angles of depression of two successive milestones on a straight level road leading from the hill are $8^{o}\ 38'$ and $16^{o}\ 6'$. How high is the hill?

28. A ship is sailing due east at the rate of 15.6 miles per hour. A lighthouse is observed due south at 8 p.m. The bearing of the same lighthouse at 10:45 p. m. was S $36^{o}\ 40'$ W. Find the distance from the lighthouse to the ship at the time of the second observation.

29. What is the angle of elevation of an inclined plane, if it rises 1 foot in a horizontal distance of 12 feet?

30. An observer in a balloon is 4 miles high. What is the distance of his visible horizon on the earth?

31. What must be the elevation of an observer in order that he may be able to see an object on the earth 25 miles away?

32. A vessel is sailing due east at a uniform rate of speed. At 7 p.m., a lighthouse is observed bearing due north, 10.4 miles distant; and at 7:30 p.m. it bears $17^{o}45'$ west of north. Find the rate of sailing of the vessel and the bearing of the lighthouse at 10 p. m.

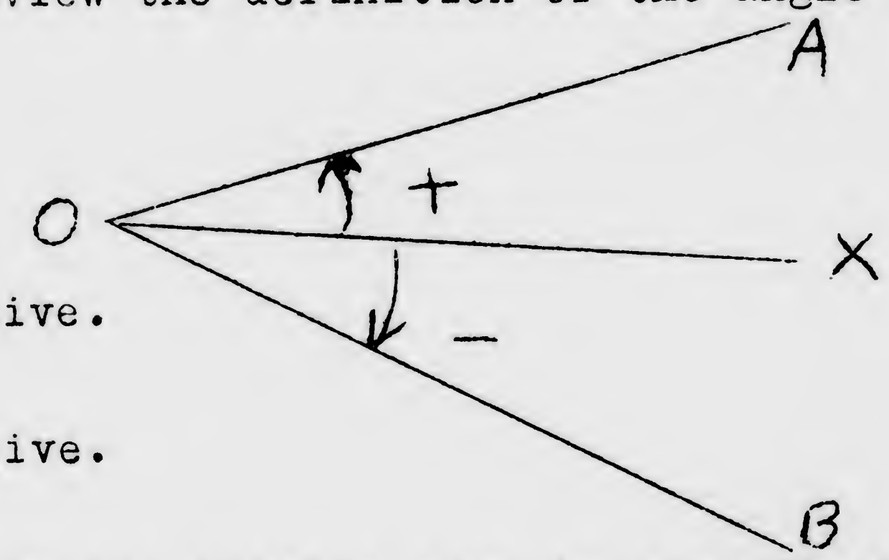
33. A railroad having eighty feet right of way cuts through a farmer's rectangular field at an angle of 27^{o} and 400 feet from a corner. Find the number of square rods occupied by the right of way and the assessed damage, if the land is valued at $300 an acre.

TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

We have so far defined the trigonometric functions of acute angles only. These definitions are sufficient for the solutions of a right triangle. Many problems, however, require the consideration of obtuse angles, and others involve angles greater than 180° or 360°. We are, therefore, led to consider the functions of angles of any size.

Positive and Negative Angles. We learned in algebra to designate opposite qualities by the words "positive" and "negative". Thus if we call a force acting upward positive, one acting downward is negative. If distances to the right of a starting point are considered positive, then distances to the left are negative. If motion in one direction is considered positive, then motion in the opposite direction is negative.

With respect to angles, it is agreed to consider an angle positive, if it is formed by rotating a line counterclockwise, that is, in the direction opposite to that taken by the hands of a clock. The angle is considered negative if the rotating line moves clockwise. (Review the definition of the angle on page one).

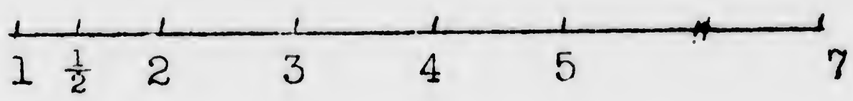


Angle X O A is positive.

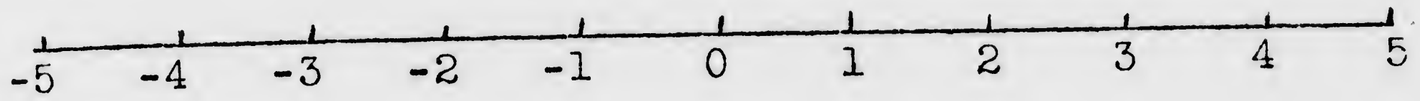
Angle X O B is negative.

Observe that the initial line is read first. A circular arrow between the two lines with its head on the terminal line indicates the direction of rotation and the magnitude of the angle.

Scales. In mathematics, we frequently speak of the arithmetical scale and the algebraic scale. The arithmetical scale begins at zero and extends indefinitely in one direction including all the positive real numbers.



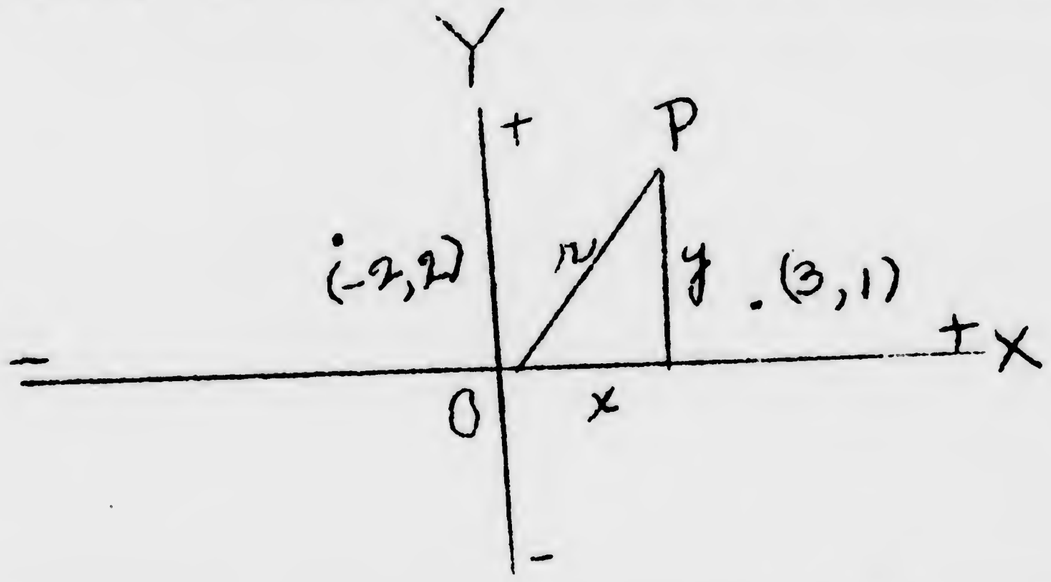
The algebraic scale extends indefinitely in opposite direction from zero, with the positive numbers to the right and the negative numbers to the left.



Rectangular Coordinates. Two algebraic scales intersecting at the zero points at right angles to each other may be taken as lines of reference or axes. We shall usually think of one of them as horizontal and call it the x-axis, and call the other the y-axis. The point of intersection O is called the origin.

To locate a point in the plane it is merely necessary to give its distance to the right or left of the y-axis, and its distance above or below the x-axis. The distance of any point from the y-axis is called the abscissa of the point or its x. Its distance from the x-axis is called the ordinate of the point or its y. The abscissa and ordinate of a point together are called its coordinates.

Abscissas of all points to the right of the y-axis are positive, and of those to the left are negative. Ordinates above the x-axis are positive, and below the x-axis are negative.



Plotting a Point. A point is designated by writing its abscissa and ordinate in a parenthesis with the abscissa first and the ordinate second. Thus (3,1) means the point whose abscissa is 3 and whose ordinate is 1. In the same way (-2,2) means the point whose abscissa is -2, and whose ordinate is + 2, and so on. To plot a point is to locate it when its x-distance and y-distance are known.

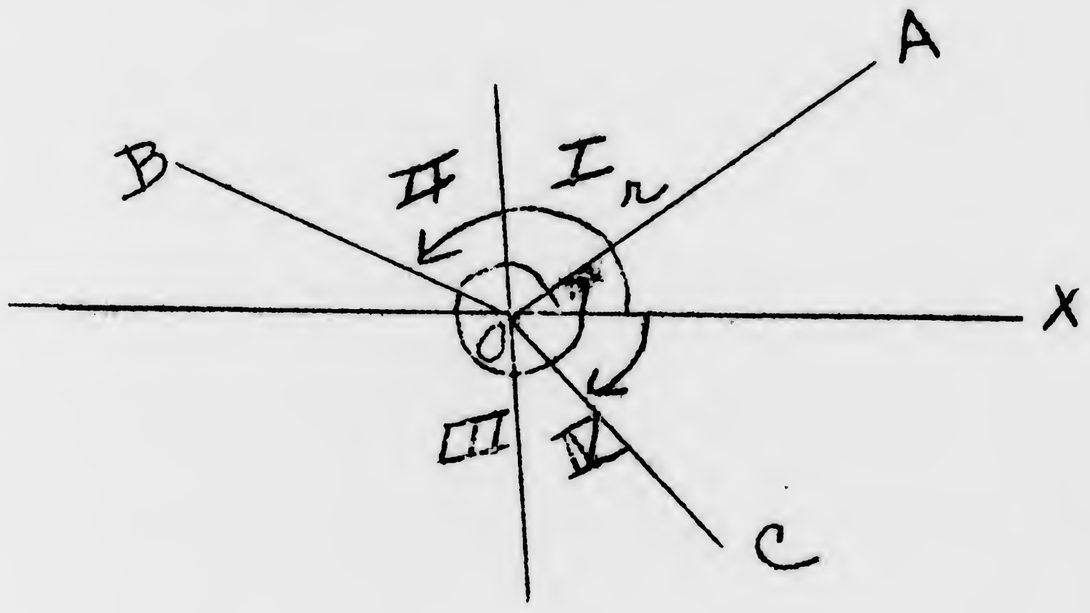
Distance from the Origin. The distance of a point from the origin is called the radius vector of that point and is denoted by r. The radius vector of a point forms with abscissa and ordinate of the point a right triangle of which the radius is the hypotenuse. Obviously the distance of a point from the origin is expressed by (see figure)

$$r = +\sqrt{x^2 + y^2}$$

The distance is always regarded as positive.

The Four Quadrants. The axes divide the plane into four parts called quadrants. These are numbered first, second, third and fourth quadrants starting from the positive side of the x-axis and going counterclockwise.

It is customary to make the initial side of an angle coincide with the positive direction of the x-axis. The angle is then said to be an angle of that quadrant in which its terminal side lies. If the terminal side falls on one of the axes, the angle may belong to either one of the two quadrants upon whose boundaries it lies.



Angle X O A is in the first quadrant.

Angle X O B is in the second quadrant.

Angle X O C is in the fourth quadrant.

Exercises.

1. Plot the point whose abscissa is 3 and whose ordinate is - 4. Find its distance from the origin.

2. Plot the following points and find their distances from the origin:

   (2, -3) , (-2, 1) (-3,4), (0,-3), (4, 0),

   $(2, \sqrt{5})$, $(2\frac{1}{2}, 5)$.

3. The abscissa of a point is 4. Is the point definitely located? Why?

4. The y of a point is -2. Is the point determined? Why?

5. Where are all the points for which x = 3? x = -1? y = 5? x = 0? y = 0?

6. What are the coordinates of the origin?

7. Compare the process of plotting points with the numbering of houses in a city.

8. Compare the process of plotting points with that of locating a place on the map by means of its longitude and latitude.

9. A point moves to the right 3 inches, up 5 inches, then to the right 4 inches; how far is it from the starting point?

10. Construct the following angles and name the quadrants in which they lie:

    40°, 75°, 120°, 150°, 225°, 330°, -60°, -90°, -400°,

    $+600^\circ$, -800°.

11. Draw the angle 256°, and give the value of a negative angle having the same terminal side. How many such angles are there?

12. Draw the terminal side of 392° and give the values of one positive angle which has the same terminal side. How many such angles are there?

13. Plot the points whose coordinates are given. Find the length of the radius vector of each. Find by measurements the angle which the radius vector makes with the positive direction of the x-axis.

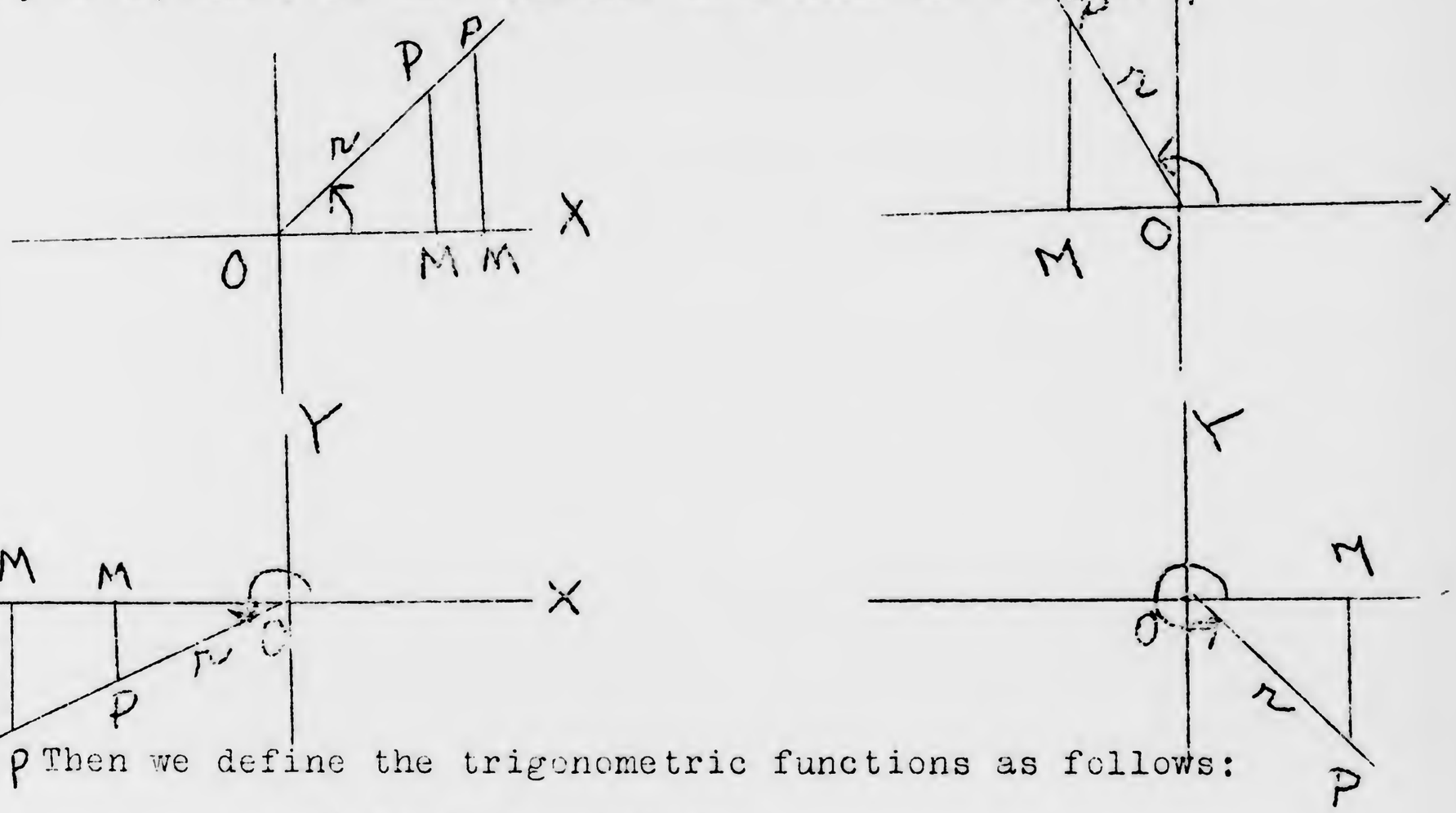
    (6,8), (-3,4), (-6,-1), (2.4, -3.7)

By means of a protractor construct:

14. $23^\circ + 105^\circ + (-60^\circ)$.
15. $-160^\circ + 80^\circ - 25^\circ$.
16. $135^\circ - 60^\circ + 80^\circ - 250^\circ$.
17. In what quadrant is the abscissa positive and the ordinate negative? Abscissa negative and the ordinate positive? Both negative? Both positive?

Functions of Any Angle. We are now ready to give the definitions of the functions of a general angle.

Let O X be the initial side and O P the terminal side of any angle A. Let P be any point on the terminal line. From P drop a perpendicular P M to the x-axis. Then O M is the x of P, M P is the y of P, and r is the distance of P from the origin.



Then we define the trigonometric functions as follows:

$$\sin A = \frac{y}{r} = \frac{\text{ordinate}}{\text{distance}}$$

$$\cos A = \frac{x}{r} = \frac{\text{abscissa}}{\text{distance}}$$

$$\tan A = \frac{y}{x} = \frac{\text{ordinate}}{\text{abscissa}}$$

$$\cot A = \frac{x}{y} = \frac{\text{abscissa}}{\text{ordinate}}$$

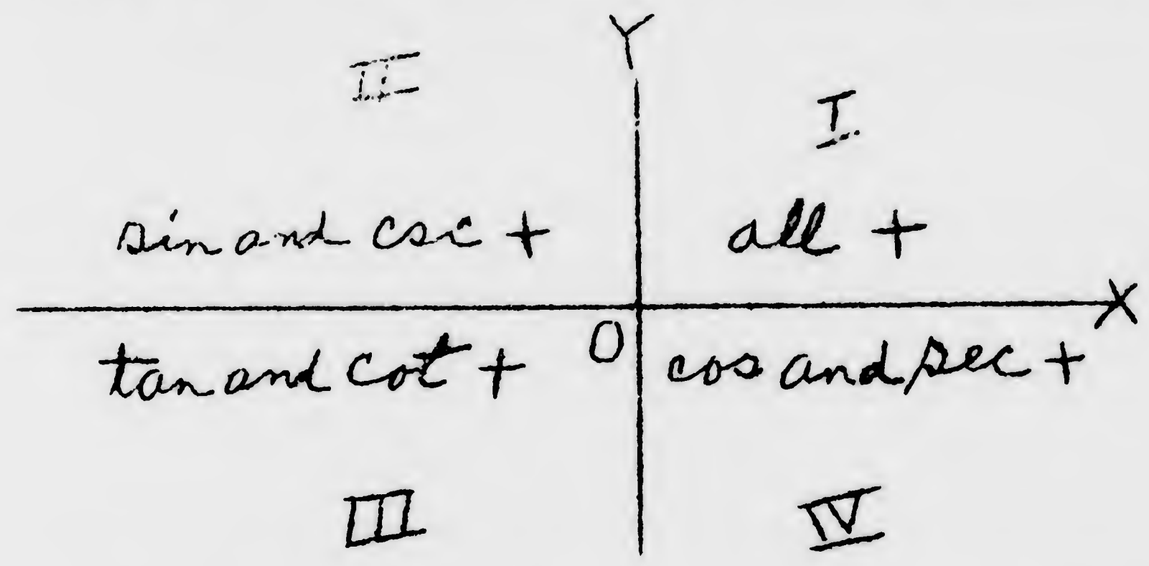
$$\sec A = \frac{r}{x} = \frac{\text{distance}}{\text{abscissa}}$$

$$\csc A = \frac{r}{y} = \frac{\text{distance}}{\text{ordinate}}$$

From a property of similar triangles it is seen that these ratios are entirely independent of the position of the point P on the terminal line, i. e. independent of the lengths of the abscissa, ordinate, and distance. They depend solely upon the size of the angle, and are, therefore, <u>functions of the angle</u>. Hence, for every angle there is one and only one value of each function. These functions are called circular, or trigonometric functions.

It is obvious that these general definitions are in accord with the special definitions of the functions of an acute angle. For if an acute angle is placed in the first quadrant, the adjacent side, the opposite side, and the hypotenuse are simply, the x, y, and r of the point P.

<u>Signs of the Functions.</u> The signs of the functions of any angle depend upon the signs of the abscissa, the ordinate, and the distance. Each function is positive intwo quadrants and negative in the other two. No rules are necessary to find this, if you construct the angle (or visulaize its position) and know the definitions thoroughly. This diagram may aid the memory.

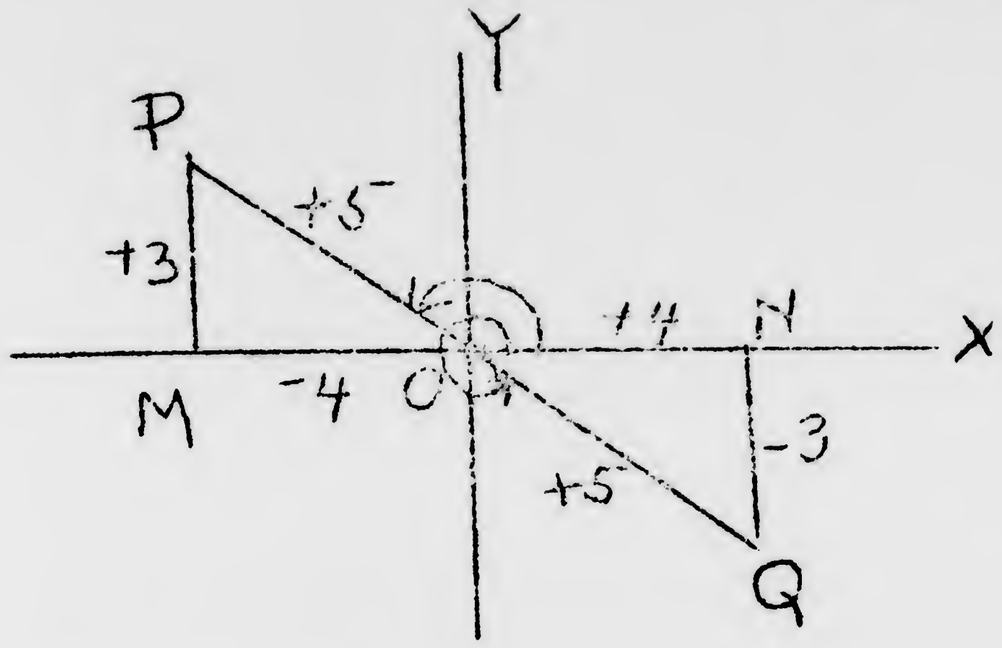


<u>Finding All Functions from One.</u> If any function of an angle is given, it is possible to construct the angle and to find the values of the other functions.

1. Given $\tan A = -\frac{3}{4}$; construct the angle and find the values of the other functions.

$$\tan A = \frac{y}{x} = \frac{+3}{-4} \text{ or } \frac{-3}{+4}$$

When the abscissa is negative and the ordinate positive, the angle is in the second quadrant. When the abscissa is positive and the ordinate negative, the angle is in the fourth quadrant. Hence, there are two positive angles less than 360° whose tangent is $-\frac{3}{4}$.



To construct the figure, find the point P whose abscissa is - 4, and ordinate is 3, and the point Q whose abscissa is + 4, and ordinate is - 3. From P and Q drop perpendiculars to the x-axis. Connect P and Q with the origin. Then each of the angles X O P and X O Q has its tangent = $-\frac{3}{4}$.

To find r we have, $r = \sqrt{x^2 + y^2} = +\sqrt{16+9} = +5$.

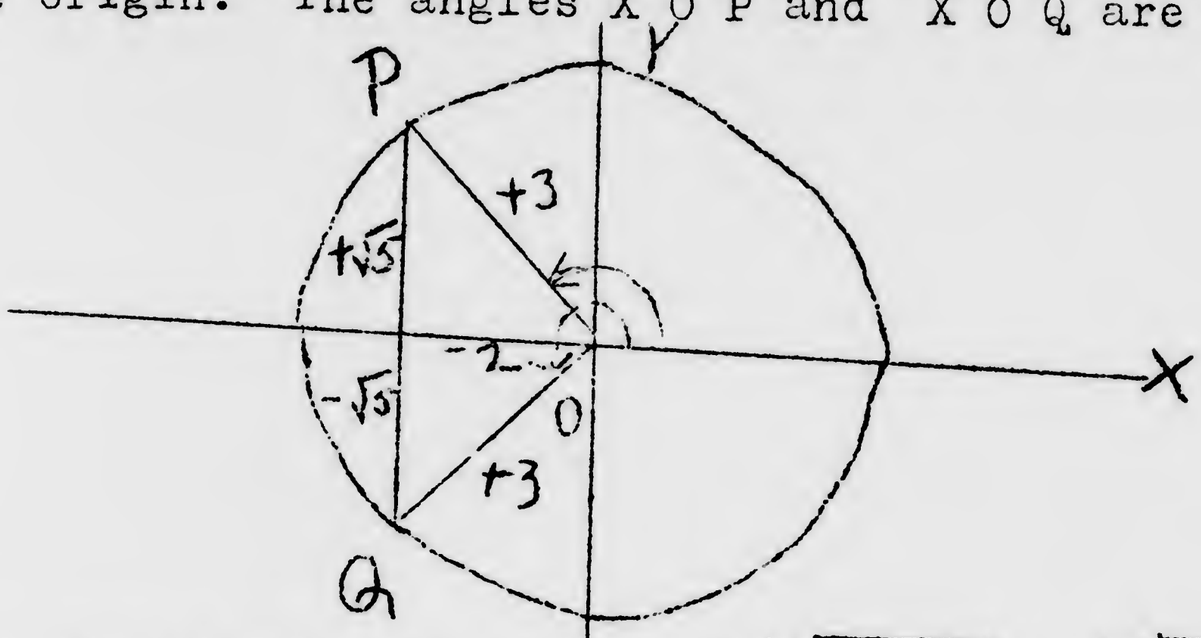
Hence by definition:

	II Quadrant	,	IV Quadrant
sin A =	$+\frac{3}{5}$	,	$-\frac{3}{5}$
cos A =	$-\frac{4}{5}$	,	$+\frac{4}{5}$
tan A =	$-\frac{3}{4}$	,	$-\frac{3}{4}$
cot A =	$-\frac{4}{3}$	,	$-\frac{4}{3}$
sec A =	$-\frac{5}{4}$	,	$+\frac{5}{4}$
csc A =	$+\frac{5}{3}$	,	$-\frac{5}{3}$

2. Construct the positive angles for which the cosine is $-\frac{2}{3}$, and find the values of the other functions of these angles.

Cos A = $\frac{x}{r} = \frac{-2}{+3}$. Since the distance is always positive, the abscissa must be negative.

Determine the points whose distance is 3 and abscissa - 2. Describe a circle with radius 3; then erect a perpendicular at x = - 2. Call the points of intersection P and Q. Connect P and Q with the origin. The angles X O P and X O Q are the angles required.



To find the length of y we have, $y = \pm\sqrt{r^2 - x^2} = \pm\sqrt{9 - 4} = \pm\sqrt{5}$. Then the coordinates of P are $(-2, \sqrt{5})$; and of Q, $(-2, -\sqrt{5})$. Hence,

	II Quadrant	,	III Quadrant
sin A =	$+\frac{\sqrt{5}}{3}$	,	$-\frac{\sqrt{5}}{3}$
cos A =	$-\frac{2}{3}$	,	$-\frac{2}{3}$
tan A =	$-\frac{\sqrt{5}}{2}$	,	$+\frac{\sqrt{5}}{2}$
cot A =	$-\frac{2}{\sqrt{5}}$	,	$+\frac{2}{\sqrt{5}}$
sec A =	$-\frac{3}{2}$	,	$-\frac{3}{2}$
csc A =	$+\frac{3}{\sqrt{5}}$	,	$-\frac{3}{\sqrt{5}}$

Remember that when the value of one function of an angle is given, there are in general two possible sets of values for the other functions, due to the fact that there are two positive angles less than 360° determined by the given function.

It is obvious that all angles terminating on the same line have identical values for their functions. Consequently, there is an unlimited number of angles above 360° whose functions are exactly the same as the functions of A. Thus the functions of 40°, 400°, 760°, -320°, -680°, etc., are identical. That is, the functions of $(n \cdot 360^\circ + A)$ are the same as the functions of A.

TIGHT BINDING

Exercise.

In what quadrants does A terminate when

1. $\sin A = -\frac{1}{3}$ 2. $\cos A = -\frac{3}{4}$. 3. $\tan A = 6$.

4. Sin A is positive and cos A is negative.

5. Tan A is positive and cos A is negative.

6. Sin A is negative and cos A is negative.

Give the signs of the functions of the following angles:

7. 520° , -25° , -120° .

8. 660° , -240° , -300° .

9. Is there an angle whose tangent is negative and whose cotangent is positive?

10. If we are told that the sine and tangent of an angle have the same sign, what can we say about the quadrant of the angle?

11. Make a diagram showing the signs of each function in the four quadrants.

12. Is there an angle whose sine is 3? Explain.

13. Is there an angle whose secant is $-\frac{1}{2}$? Explain.

14. Is there an angle whose cosine is - 2? Explain.

Construct carefully the following angles and find the values of the functions. Determine the number of degrees by measurement and also from the tables of natural functions.

15. $\sin A = \frac{1}{3}$ 16. $\cos A = -\frac{2}{5}$ 17. $\tan A = \frac{4}{3}$

18. $\cot A = -\frac{3}{5}$. 19. $\sec A = 2$ 20. $\csc A = -\frac{3}{2}$

21. $\cos A = .4$ 22. $\tan A = -3$ 23. $\sin A = \frac{a}{b}$

24. $\cos A = \frac{3}{7}$ and sin A is negative.

25. $\sin A = .7$ and tan A is negative.

26. $\tan A = 5$ and sin A is negative.

27. Find the value of cos A tan A sec A when $\sin A = .3$.

28. Find the value of $\dfrac{\sin A - \cot A + \cos A}{\sec A + \csc A - \cot A}$

When $\tan A = \frac{3}{4}$ and A is in the third quadrant.

29. Given $\cos A = .9$, find the value of $\cos^2 A + \sin^2 A$.

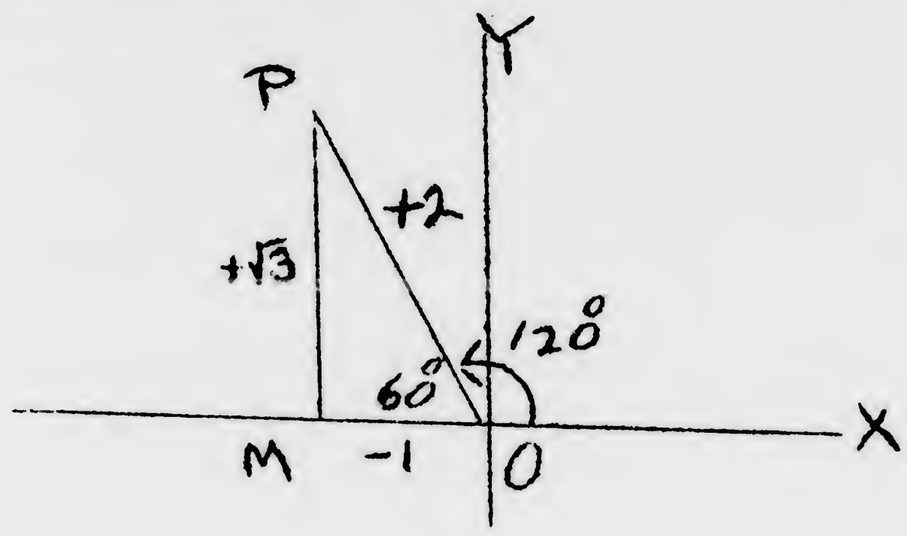
30. Given $\tan A = \frac{1}{2}$, find the value of $\dfrac{\sin A}{\cos A}$.

31. $\sin B = m$; find the values of the other functions. Are all possible values of m admissible? State a reason for your answer.

To Find the Functions When the Angle is Given. When an angle is given there is one and only one value of each function. The values of these functions can be found approximately by measurement. For mathematical work, we must consult trigonometric tables. For a few angles the functions can also be calculated exactly by geometry. Among these special angles are 30°, 45°, and 60° which have already been considered (Review them).

Functions of 120°, 135°, 150°. These angles have the same numerical ratios between the abscissa, the ordinate and the distance as the ratios between the adjacent side, the opposite side, and the hypotenuse of the angles 60°, 45° and 30° respectively.

Functions of 120°. Construct the angle 120 as in the figure.



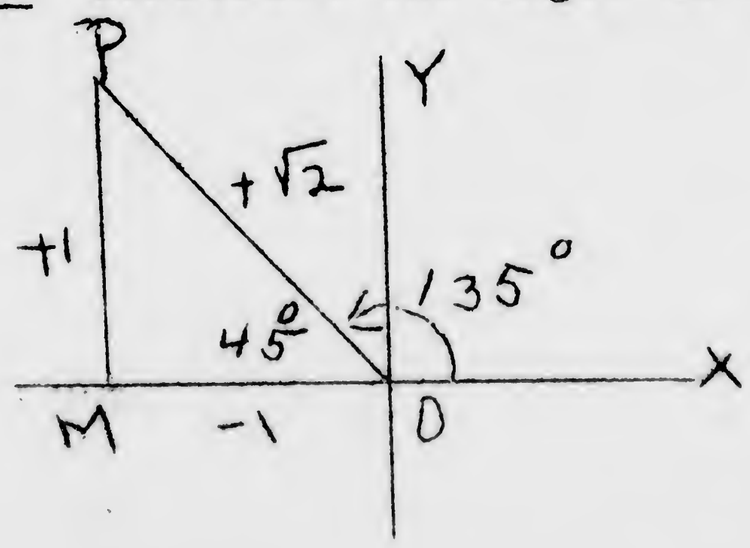
Since angle P O M is 60°, the sides of the triangle are in the ratio $-1, +\sqrt{3}$, and $+2$ respectively, and the coordinates of P are $(-1, +\sqrt{3})$. From the definitions it is evident that

$\sin 120^\circ = +\frac{\sqrt{3}}{2} = .8660$ $\quad$ $\cot 120^\circ = -\frac{1}{\sqrt{3}} = -\frac{1}{3}\sqrt{3} = .5774$

$\cos 120^\circ = -\frac{1}{2} = -.5000$ $\quad$ $\sec 120^\circ = -2.0000$

$\tan 120^\circ = -\sqrt{3} = -1.7321$ $\quad$ $\csc 120^\circ = +\frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3} = 1.1548$

Functions of 135°. Construct the angle as in the figure.



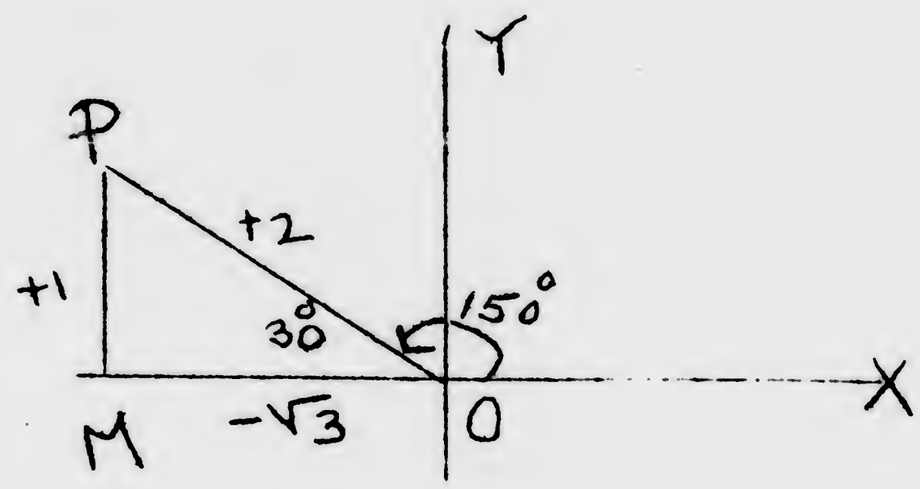
Since angle P O M is 45° , the coordinates of P are $(-1, +1)$. Hence,

$\sin 135^{\circ} = +\frac{1}{\sqrt{2}} = .7071$ $\qquad$ $\cot 135^{\circ} = -1.0000$

$\cos 135^{\circ} = -\frac{1}{\sqrt{2}} = -.7071$ $\qquad$ $\sec 135^{\circ} = -\sqrt{2} = -1.414$

$\tan 135^{\circ} = -1.0000$ $\qquad$ $\csc 135^{\circ} = +\sqrt{2} = 1.414$

Functions of 150°. Construct the angle 150° as in the figure.



Since angle P O M is 30° , the coordinates of P are $(-\sqrt{3}, 1)$. Hence,

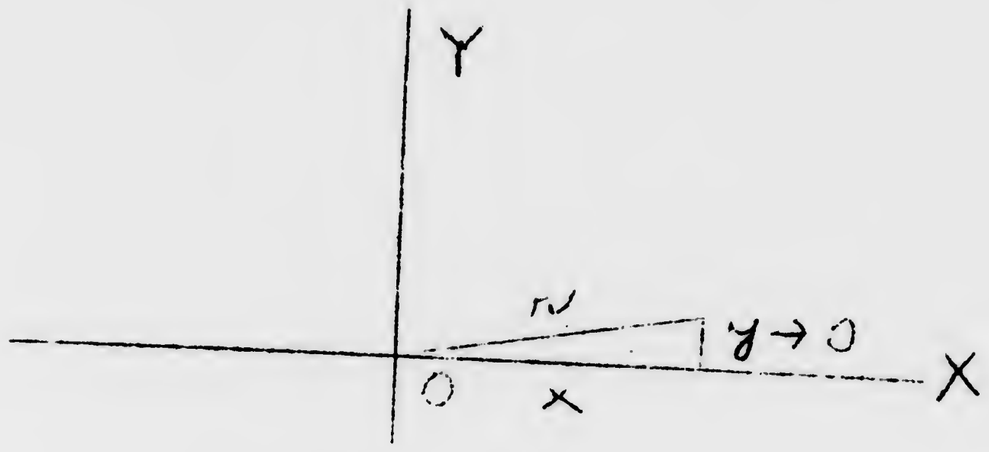
$\sin 150^{\circ} = +\frac{1}{2}$ $\qquad$ $\cot 150^{\circ} = -\sqrt{3}$

$\cos 150^{\circ} = -\frac{\sqrt{3}}{2}$ $\qquad$ $\sec 150^{\circ} = -\frac{2}{\sqrt{3}}$

$\tan 150^{\circ} = -\frac{1}{\sqrt{3}}$ $\qquad$ $\csc 150^{\circ} = +2$

Functions of 210°, 225°, 240°, 300°, 315°, 330°. These angles in the third and fourth quadrants are connected in the same way with the angles 30°, 45°, and 60° in the first quadrant. Construct the figure for each one of these angles as has been don for 120°, 135° and 150° and find the values of the functions.

Functions of 0^o . The functions of 0^o are limiting values. Take a very small angle as shown in the figure. Then as the angle approaches 0^o, y approaches 0 and x approaches r.



Then,

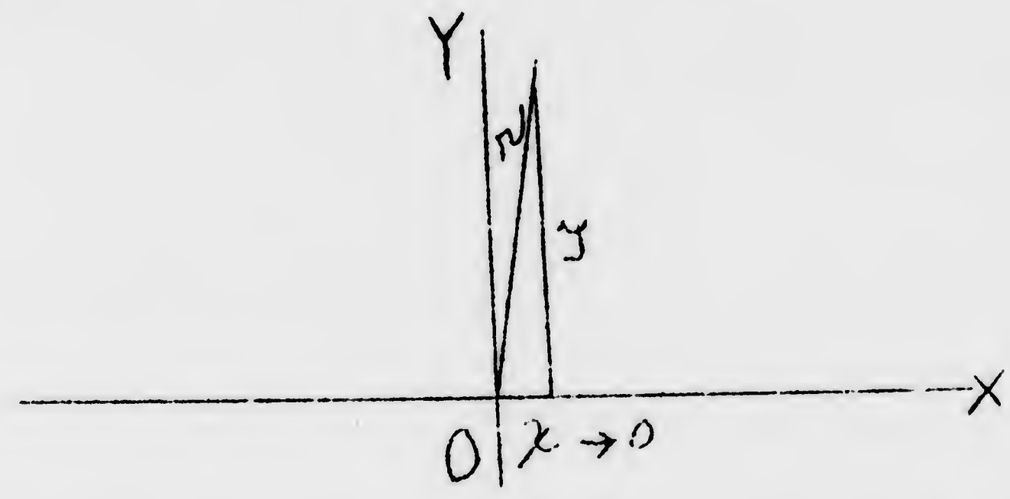
$\sin 0^o = 0$ $\qquad$ $\cot 0^o = oo$ (infinity)

$\cos 0^o = 1$ $\qquad$ $\sec 0^o = 1$

$\tan 0^o = 0$ $\qquad$ $\csc 0^o = oo$

Strictly speaking, $\cot 0^o$ and $\csc 0^o$ do not exist, as a division by zero is involved. The meaning of $\cot 0^o = oo$ is that as the angle approaches 0^o, the value of the cotangent ratio increases without limit. The same is true for $\csc 0^o$, $\tan 90^o$, $\sec 90^o$, $\cot 180^o$, $\csc 180^o$, $\tan 270^o$, $\sec 270^o$.

Functions of 90^o . Construct an angle near 90^o as in the figure. As the angle approaches 90^o, y approaches r and x approaches 0.



$\sin 90^o = 1$ $\qquad$ $\cot 90^o = 0$

$\cos 90^o = 0$ $\qquad$ $\sec 90^o = oo$

$\tan 90^o = oo$ $\qquad$ $\csc 90^o = 1$

Functions of 180°, 270°, 360°. In a similar manner, construct figures for 180°, 270° and 360° and find the values of their functions.

The special angles we have considered and the values of their functions are of such common use that it is well to learn them. They should not be memorized, but the method of construction should be thoroughly mastered.

Exercise.

1. Make a table comparing the functions of 30°, 150°, 210°, and 330°. Find the values from figures.

2. Make a table comparing the functions of 45°, 135°, 225°, and 315°.

3. Make a table comparing the functions of 60°, 120°, 240°, and 300°.

4. Make a table of the functions of 0°, 90°, 180°, 270°, and 360°.

5. Does $\cos 120^\circ = 2$ times $\cos 60^\circ$?

6. Does $\tan 135^\circ = 3 \tan 45^\circ$?

7. Does $\sin 90^\circ = 3$ times $\sin 30^\circ$?

8. Show that $\sin 120^\circ = 2 \sin 60^\circ \cdot \cos 60^\circ$.

9. Show that $\tan 240^\circ = \dfrac{2 \tan 120^\circ}{1 - \tan^2 120^\circ}$.

10. Show that $\cos 90^\circ = \cos 60^\circ \cdot \sin 30^\circ - \sin 60^\circ \cdot \sec 30^\circ$.

Find the values of the following:

11. $2 \sin 0^\circ + 3 \cos 90^\circ - 4 \sec 180^\circ$.

12. $3 \sin 90^\circ \cdot \cos 300^\circ \cdot \sec 0^\circ$.

13. $\cos 180^\circ \cdot \sin 270^\circ \cdot \tan 135^\circ \cdot \cot 225^\circ$.

14. $\dfrac{\sin^2 60^\circ - 2 \cos 180^\circ + \sec 120^\circ}{\tan 225^\circ - \cot^2 150^\circ + \csc 330^\circ}$

15. $\dfrac{\cos^2 210^\circ - \sin 270^\circ + \cot 225^\circ}{\csc 150^\circ + 3 \tan 315^\circ - \sec^2 300^\circ}$

16. Why may $\tan 90^\circ$ be considered $+\infty$ and $-\infty$?

17. Find the value of $\cos A + \sin A$ and of $\cos^2 A + \sin^2 A$ for $A = 60^\circ$, 120°, 180°, 210°, 325°.

18. Find two positive angles less than 360° that satisfy the following conditions:

$\cos A = -\frac{1}{2}$, $\tan A = 1$, $\sec A = 2$

$\sin A = -\frac{1}{\sqrt{2}}$, $\cot A = -\sqrt{3}$, $\csc A = -\sqrt{2}$.

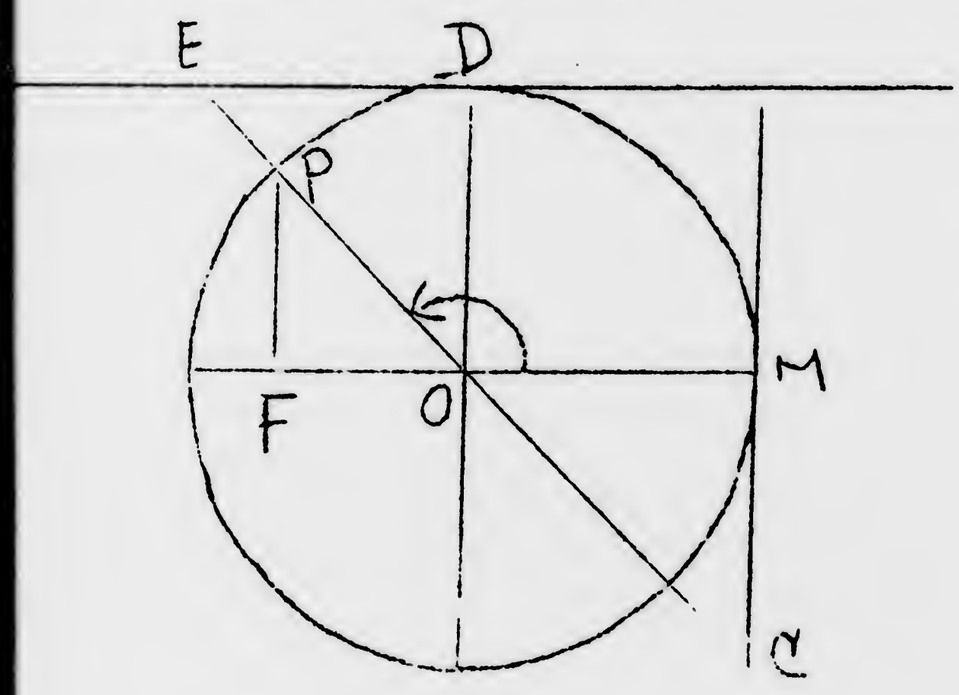
19. In each of the following find the four smallest angles that satisfy the given conditions:

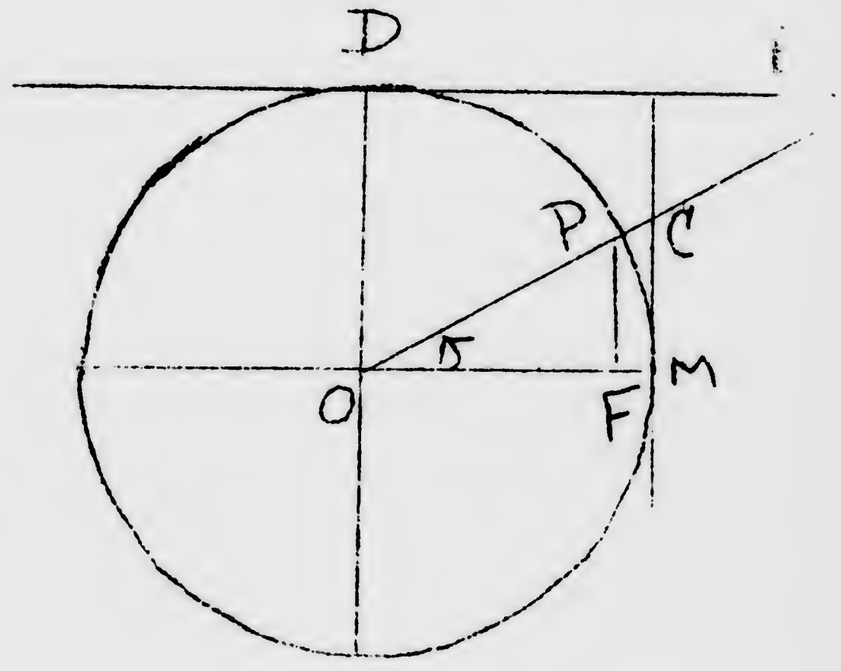
$\sin A = -\frac{\sqrt{3}}{2}$, $\tan A = \frac{1}{\sqrt{3}}$, $\sec A = -\frac{2}{\sqrt{3}}$

$\cos A = -\frac{1}{\sqrt{2}}$, $\cot A = -1$, $\csc A = 2$

20. Construct and find the ratios of 420°, 495°, 765°, 690°, -405°, -930°.

<u>Representation of the Functions by Lines.</u> Although the trigonometric functions are not lines but abstract numbers, it is interesting to see how they may be represented by lines in a circle of radius unity.





In each figure

$\sin A = \frac{F\,P}{O\,P} = F\,P$ $\quad\quad$ $\cot A = \frac{O\,F}{F\,P} = \frac{D\,E}{O\,D} = D\,E$

$\cos A = \frac{O\,F}{O\,P} = O\,F$ $\quad\quad$ $\sec A = \frac{C\,P}{O\,F} = \frac{O\,C}{O\,M} = O\,C$

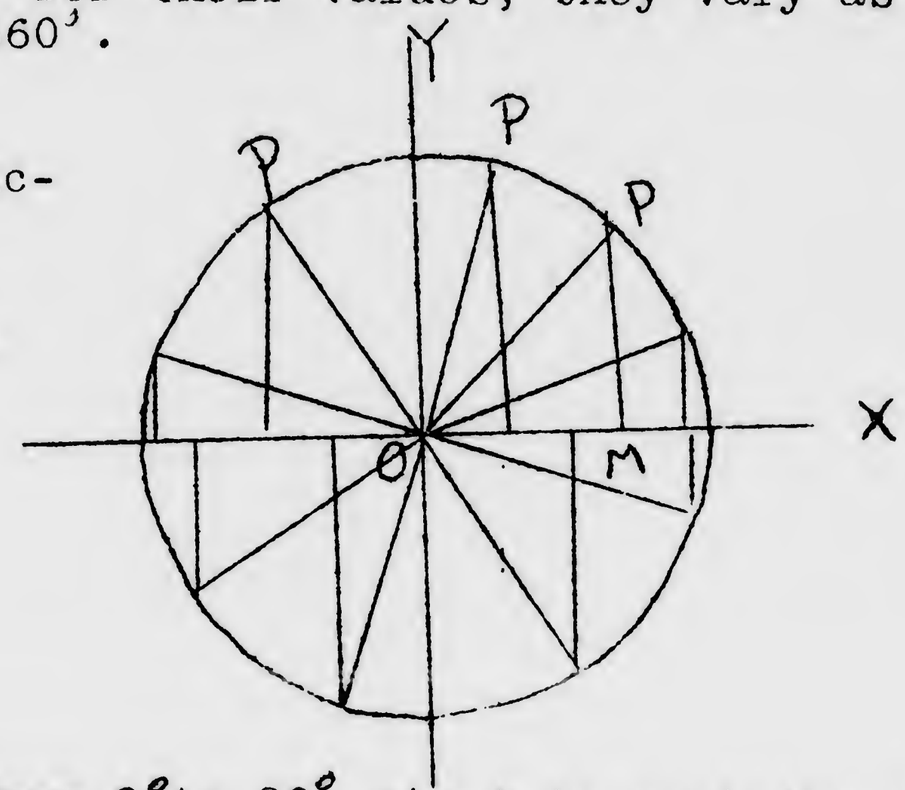
$\tan A = \frac{M\,C}{O\,M} = M\,C$ $\quad\quad$ $\csc A = \frac{O\,P}{F\,P} = \frac{O\,E}{O\,D} = O\,E$

Observe that the lines represent the functions in sign as well as in magnitude.

The student should represent the functions by lines when the angle is in the third quadrant, also in the fourth.

Variations in the Functions. Since the trigonometric functions depend upon the angle for their values, they vary as the angle increases from 0° to 360°.

The changes in the functions can be traced by observing the changes in the abscissa and ordinate of P.



As angle A increases from 0° to 90°, sin A is positive and increases from 0 to +1.

As A increases from 90° to 180°, sin A is positive and decreases from 1 to 0.

As A increases from 180° to 270°, sin A is negative and decreases from 0 to -1.

As A increases from 270° to 360°, sin A is negative and decreases from -1 to 0.

Periodicity. As A increases from 360° to 720°, sin A passes through the same changes and in the same order, and this it does during each successive revolution. That is, the sine is a periodic function and its period is 360°.

The student should learn to describe the variation in each function as the angle increases from 0° to 360°. It will be seen that the cosine, secant, and cosecant are periodic functions and that each of them has a period of 360°. The tangent and cotangent repeat themselves every 180°.

Many physical quantities vary periodically in the same way as the trigonometric functions. The rotation of the earth and its revolution, the phases of the moon, alternating currents, wa motion, the vibrations of a violin string, the swing of a pendu etc., are periodic in nature. Such periodicity may be represent mathematically by means of the trigonometric functions. This is another important use of trigonometry.

Important Observations.

1. In the first quadrant as the angle increases from 0° to 90°, the sine, the tangent, and the secant increases, while the functions decrease.

2. The functions change signs as they pass through the values of 0 and oo.

3. The sine and cosine can never have values greater than $+1$ or less than -1. The secant and cosecant have values from $+1$ to $+oo$, and have no values between -1 and $+1$. The tangent and cotangent may have any value from $-oo$ to $+oo$.

Exercise.

1. For what angles less than 90° is the sine greater than the cosine? Less than the cosine? Equal to the cosine?

2. How do the tangents of the angles from 0° to 45° differ from the tangents of the angles from 45° to 90°?

3. How often does the sine repeat itself? The tangent?

4. Describe the variation in cos A as A increases from 0° to 360°.

5. Describe the variation in tan A as A increases from 0° to 360°. Do the same with the remaining functions.

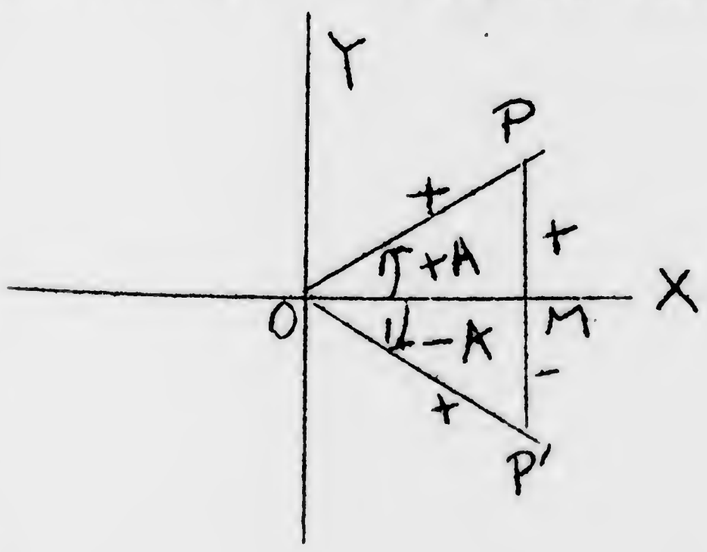
6. Compare sin A and tan A for angles less than 5°; for angles between 85° and 90°.

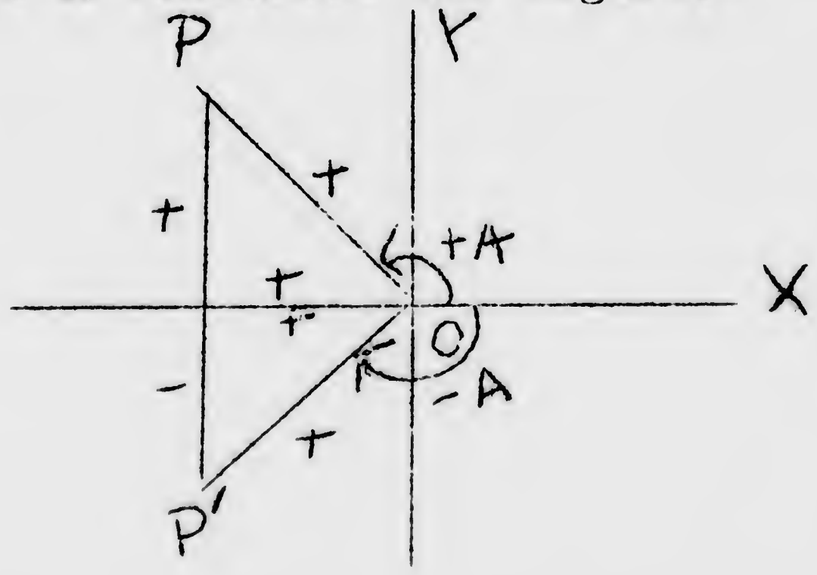
7. Compare the variation in the sine of angles from 0° to 5° with angles from 85° to 90°. Do the same for the cosine and for the tangent.

Related Angles.

We shall now learn a method for finding the functions of any angle, in terms of an angle less than 90° or less than 45°.

Functions of Negative Angles. In order to express the functions of (-A) in terms of A construct the figures.





TIGHT BINDING

In either figure the triangles M O P and M O P' are equal. $x' = x$, $y' = y$, $r' = r$. It is evident that the abscissas have the same sign and the ordinates have opposite signs. Hence,

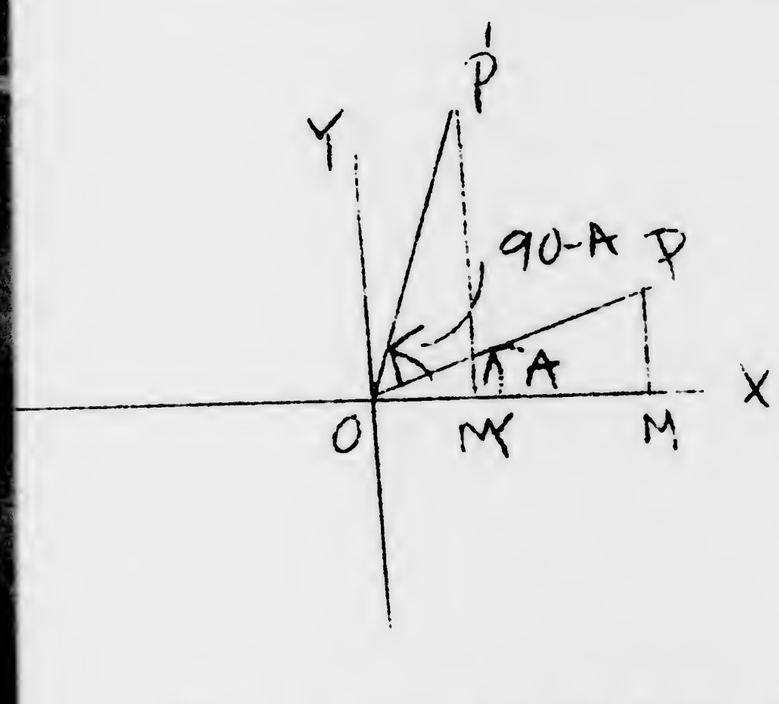
$\sin(-A) = -\sin A$ $\qquad$ $\cot(-A) = -\cot A$

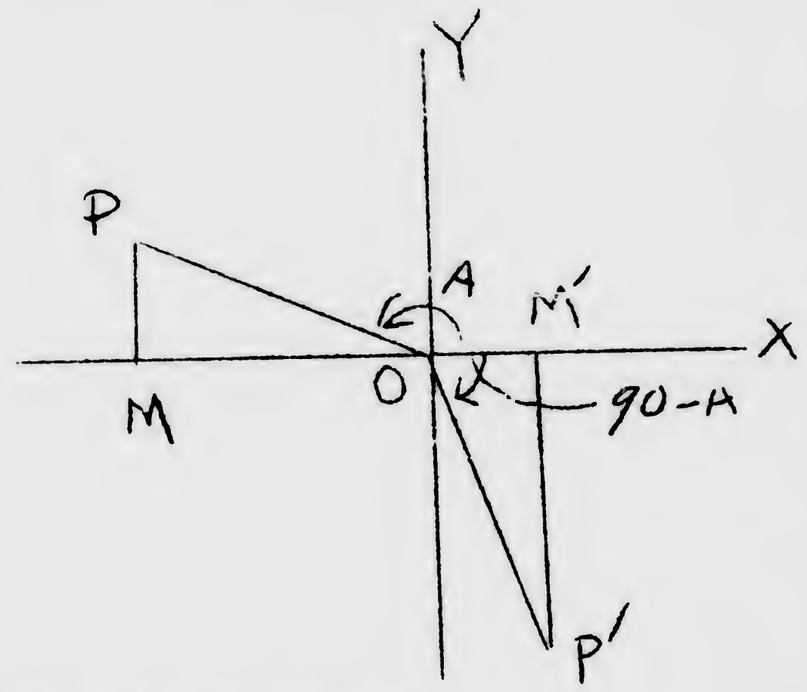
$\cos(-A) = +\cos A$ $\qquad$ $\sec(-A) = +\sec A$

$\tan(-A) = -\tan A$ $\qquad$ $\csc(-A) = -\csc A$

Law. Any function of a negative angle is equal to the same function of an equal positive angle, the sine being changed in all cases except for the cosine and secant.

Functions of $(90^\circ - A)$ and $(90^\circ + A)$ in terms of A.





The triangles M O P and M' O P' are equal.

$x' = y$, $y' = x$, $r' = x$. Hence

$\sin(90^\circ - A) = \cos A$ $\qquad$ $\cot(90^\circ - A) = \tan A$

$\cos(90^\circ - A) = \sin A$ $\qquad$ $\sec(90^\circ - A) = \csc A$

$\tan(90^\circ - A) = \cot A$ $\qquad$ $\csc(90^\circ - A) = \sec A$

Law. The functions of an angle are equal to the corresponding co-functions of its complement.

In a similar way draw appropriate figures to show that the functions of $(90 + A)$ are related to A as follows:

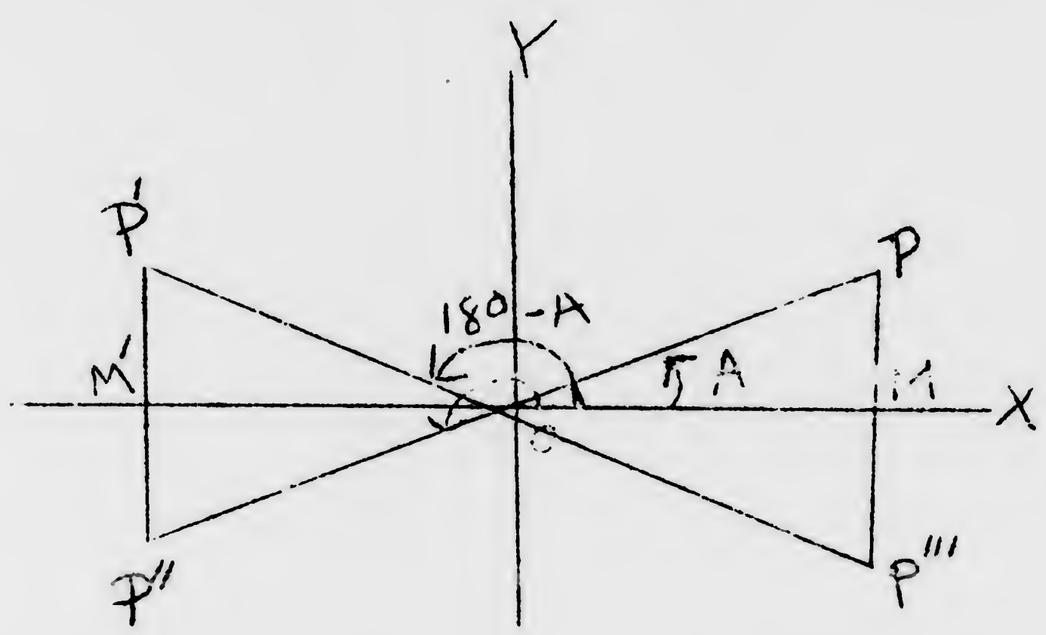
$\sin(90^\circ + A) = \cos A$ $\qquad$ $\cot(90^\circ + A) = -\tan A$

$\cos(90^\circ + A) = -\sin A$ $\qquad$ $\sec(90^\circ + A) = -\csc A$

$\tan(90^\circ + A) = -\cot A$ $\qquad$ $\csc(90^\circ + A) = \sec A$

These results show that the law for the functions of $90^\circ - A$ hold also for $90^\circ + A$, but the sign is changed except for the sine and cosecant.

Functions of $(180^\circ - A)$ and $(180^\circ + A)$ in terms of A.



The triangle M O P and M' O P' are equal.

$$x' = -x \ , \ y' = y, \quad r' = r.$$

$$\sin (180^\circ - A) = \sin A \qquad \cot (180^\circ - A) = -\cot A$$

$$\cos (180^\circ - A) = -\cos A \qquad \sec (180^\circ - A) = -\sec A$$

$$\tan (180^\circ - A) = -\tan A \qquad \csc (180^\circ - A) = \csc A$$

Law. The function of anyangle is equal to the same functi of its supplement, the sign being changed in all except the sine and cosecant.

The functions of $(180^\circ \pm A)$ are the same as the functions of A, regard being had for the algebraic sign.

General Rule. A complete investigation would show the following rules to be true:

(1) When n is even, any function of $(n.\ 90^\circ \pm A)$ is numerically equal to the same function of A.

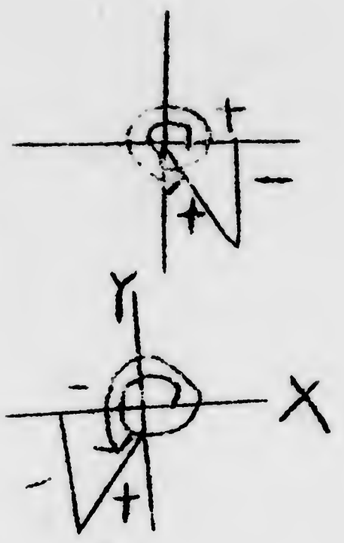
(2) When n is odd, any function of $(n.\ 90^\circ \pm A)$ is numerically equal to the co-function of A.

(3) The sign of the resulting function of A is the same as the sign of the original function of $(n.90^\circ \pm A)$ as determined by the quadrant of $(n.90^\circ \pm A)$.

TIGHT BINDING

To illustrate:

1. $\sin 650^\circ = \sin (7.90^\circ + 20^\circ) = -\cos 20^\circ$

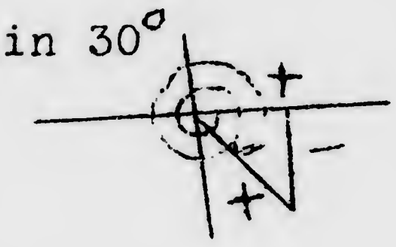


2. $\tan 590^\circ = \tan (6.90^\circ + 50^\circ) = +\tan 50^\circ = \cot 40^\circ$

or

$\tan 590^\circ = \tan (7.90^\circ - 40^\circ) = \cot 40^\circ$

3. $\cos (-780^\circ) = \cos(-8.90^\circ - 60^\circ) = +\cos 60^\circ = \sin 30^\circ$



or

$\cos (-780^\circ) = \cos (-9.90^\circ + 30^\circ) = +\sin 30^\circ.$

4. $\sec (-1155^\circ) = \sec (-12 \cdot 90^\circ - 75^\circ) = +\sec 75^\circ = \csc 15^\circ,$

or

$\sec(-1155^\circ) = \sec(-3 \cdot 360^\circ - 75^\circ) = \sec(-75^\circ) = \sec 75^\circ = \csc 15^\circ.$

Exercise

1. Compare the trigonometric functions of any angle with those of its complement; with those of its supplement.

2. How would you find the values of the functions of an obtuse angle? Find the sine, cosine, and tangent of 145°.

3. Find the sine, cosine and tangent of 305°, 94°, 156°, 24°, and $261^\circ 16'.6$.

4. Find without the trigonometric tables the values of:

   $\sin (-1260^\circ)$, $\sec 570^\circ$, $\tan 1665^\circ$.

   $\cot 1500^\circ$, $\cos (-1125^\circ)$, $\tan (-930^\circ)$.

5. Show by drawing a figure that $\cos 310^\circ = -\cos (-130^\circ)$, and that $\sin 310^\circ = -\sin 50^\circ$.

6. Express as functions of an acute angle:

   $\sin 845^\circ$, $\tan 480^\circ 20'$, $\cos (-500^\circ 40')$

   $\sec 1326^\circ 30'$, $\cot(-978^\circ 15')$, $\sin (-1184^\circ 50')$

7. Express as functions of an acute angle less than 45 ,

   $\tan 250^\circ$, $\sin (-105^\circ)$, $\cot (-400^\circ)$

   $\csc 560^\circ 20'$, $\cos 700^\circ$, $\sec(-620^\circ)$.

TIGHT BINDING

Relations Between the Functions.

There are certain fundamental relations between the functions of any angle that are of great importance. The proofs and the formulas should be thoroughly learned because they are frequently needed.

By definition we have,

$$\sin A = \frac{y}{r} \qquad \csc A = \frac{r}{y}$$

$$\cos A = \frac{x}{r} \qquad \sec A = \frac{r}{x}$$

$$\tan A = \frac{y}{x} \qquad \cot A = \frac{x}{y}$$

Hence we have three pairs of reciprocals:

$$\sin A \cdot \csc A = 1 \quad \text{or} \quad \sin A = \frac{1}{\csc A} \quad \text{or} \quad \csc A = \frac{1}{\sin A}$$

$$\cos A \cdot \sec A = 1 \quad \text{or} \quad \cos A = \frac{1}{\sec A} \quad \text{or} \quad \sec A = \frac{1}{\cos A}$$

$$\tan A \cdot \cot A = 1 \quad \text{or} \quad \tan A = \frac{1}{\cot A} \quad \text{or} \quad \cot A = \frac{1}{\tan A}$$

No matter in what quadrant the angle may be, the following relation is always true,

$$x^2 + y^2 = r^2$$

If we divide both members of this equation by r^2, we find,

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1.$$

But $\frac{x}{r} = \cos A$, and $\frac{y}{r} = \sin A$.

Therefore, $\sin^2 A + \cos^2 A = 1$.

If we divide both members of the same equation by x^2, we have,

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$$

or

$$1 + \tan^2 A = \sec^2 A$$

Again, if we divide both members by y^2, we have,

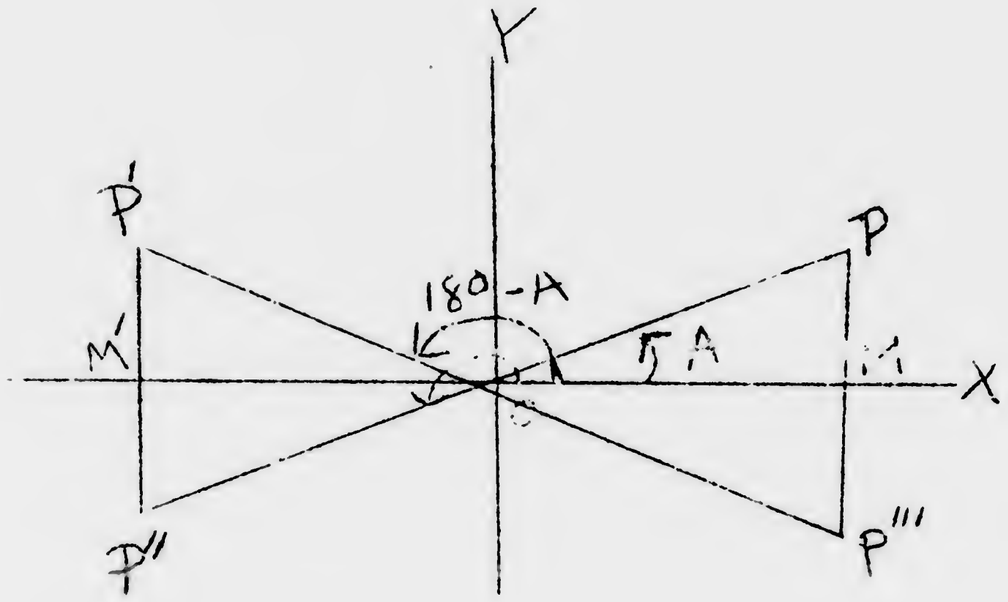
$$\left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2$$

or

$$\cot^2 A + 1 = \csc^2 A.$$

These results show that the law for the functions of $90^\circ - A$ hold also for $90^\circ + A$, but the sign is changed except for the sine and cosecant.

Functions of $(180^\circ - A)$ and $(180^\circ + A)$ in terms of A.



The triangle M O P and M' O P' are equal.

$$x' = -x, \quad y' = y, \quad r' = r.$$

$\sin (180^\circ - A) = \sin A$ $\quad\quad$ $\cot (180^\circ - A) = -\cot A$

$\cos (180^\circ - A) = -\cos A$ $\quad\quad$ $\sec (180^\circ - A) = -\sec A$

$\tan (180^\circ - A) = -\tan A$ $\quad\quad$ $\csc (180^\circ - A) = \csc A$

Law. The function of anyangle is equal to the same functi of its supplement, the sign being changed in all except the sine and cosecant.

The functions of $(180^\circ \pm A)$ are the same as the functions of A, regard being had for the algebraic sign.

General Rule. A complete investigation would show the following rules to be true:

(1) When n is even, any function of $(n.\ 90^\circ \pm A)$ is numerically equal to the same function of A.

(2) When n is odd, any function of $(n.\ 90^\circ \pm A)$ is numerically equal to the co-function of A.

(3) The sign of the resulting function of A is the same as the sign of the original function of $(n.90^\circ \pm A)$ as determined by the quadrant of $(n.90^\circ \pm A)$.

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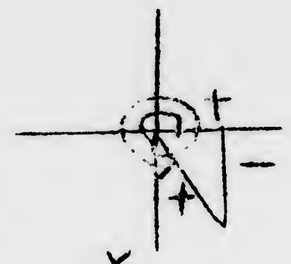
To illustrate:

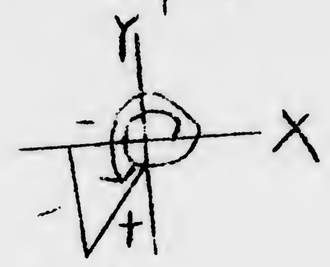
1. $\sin 650^\circ = \sin (7.90^\circ + 20^\circ) = -\cos 20^\circ$



2. $\tan 590^\circ = \tan (6.90^\circ + 50^\circ) = +\tan 50^\circ = \cot 40^\circ$

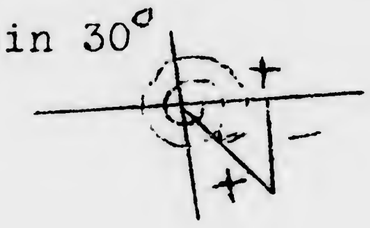
or

$\tan 590^\circ = \tan (7.90^\circ - 40^\circ) = \cot 40^\circ$



3. $\cos (-780^\circ) = \cos(-8.90^\circ - 60^\circ) = +\cos 60^\circ = \sin 30^\circ$

or



$\cos (-780^\circ) = \cos (-9.90^\circ + 30^\circ) = +\sin 30^\circ.$

4. $\sec (-1155^\circ) = \sec (-12 \cdot 90^\circ - 75^\circ) = +\sec 75^\circ = \csc 15^\circ,$

or

$\sec(-1155^\circ) = \sec(-3 \cdot 360^\circ - 75^\circ) = \sec(-75^\circ) = \sec 75^\circ = \csc 15^\circ.$

Exercise

1. Compare the trigonometric functions of any angle with those of its complement; with those of its supplement.

2. How would you find the values of the functions of an obtuse angle? Find the sine, cosine, and tangent of 145°.

3. Find the sine, cosine and tangent of 305°, 94°, 156°, 24°, and $261^\circ 16'.6$.

4. Find without the trigonometric tables the values of:

   $\sin (-1260^\circ)$, $\sec 570^\circ$, $\tan 1665^\circ$.

   $\cot 1500^\circ$, $\cos (-1125^\circ)$, $\tan (-930^\circ)$.

5. Show by drawing a figure that $\cos 310^\circ = -\cos (-130^\circ)$, and that $\sin 310^\circ = -\sin 50^\circ$.

6. Express as functions of an acute angle:

   $\sin 845^\circ$, $\tan 480^\circ 20'$, $\cos (-500^\circ 40')$

   $\sec 1326^\circ 30'$, $\cot(-978^\circ 15')$ $\sin (-1184^\circ 50')$

7. Express as functions of an acute angle less than 45 ,

   $\tan 250^\circ$, $\sin (-105^\circ)$, $\cot (-400^\circ)$

   $\csc 560^\circ 20'$, $\cos 700^\circ$, $\sec(-620^\circ)$.

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Relations Between the Functions.

There are certain fundamental relations between the functions of any angle that are of great importance. The proofs and the formulas should be thoroughly learned because they are frequently needed.

By definition we have,

$$\sin A = \frac{y}{r} \qquad \csc A = \frac{r}{y}$$

$$\cos A = \frac{x}{r} \qquad \sec A = \frac{r}{x}$$

$$\tan A = \frac{y}{x} \qquad \cot A = \frac{x}{y}$$

Hence we have three pairs of reciprocals:

$$\sin A \cdot \csc A = 1 \quad \text{or} \quad \sin A = \frac{1}{\csc A} \quad \text{or} \quad \csc A = \frac{1}{\sin A}$$

$$\cos A \cdot \sec A = 1 \quad \text{or} \quad \cos A = \frac{1}{\sec A} \quad \text{or} \quad \sec A = \frac{1}{\cos A}$$

$$\tan A \cdot \cot A = 1 \quad \text{or} \quad \tan A = \frac{1}{\cot A} \quad \text{or} \quad \cot A = \frac{1}{\tan A}$$

No matter in what quadrant the angle may be, the following relation is always true,

$$x^2 + y^2 = r^2$$

If we divide both members of this equation by r^2, we find,

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1.$$

But $\frac{x}{r} = \cos A$, and $\frac{y}{r} = \sin A$.

Therefore, $\sin^2 A + \cos^2 A = 1$.

If we divide both members of the same equation by x^2, we have,

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$$

or

$$1 + \tan^2 A = \sec^2 A$$

Again, if we divide both members by y^2, we have,

$$\left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2$$

or

$$\cot^2 A + 1 = \csc^2 A.$$

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We have also $$\tan A = \frac{y}{x} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{\sin A}{\cos A}$$

and $$\cot A = \frac{1}{\tan A} = \frac{\cos A}{\sin A} .$$

Summary of Fundamental Relations.

$\sin A \cdot \csc A = 1$, $\tan A \cdot \cot A = 1$, $\cos A \cdot \sec A = 1$.

$\sin^2 A + \cos^2 A = 1$, $1 + \tan^2 A = \sec^2 A$, $1 + \cot^2 A = \csc^2 A$.

$\tan A = \frac{\sin A}{\cos A}$ $\cot A = \frac{\cos A}{\sin A}$

From these fundamental relations many others may be derived. To illustrate:

$\sin A = \frac{1}{\csc A}$, $\sin^2 A = 1 - \cos^2 A$, $\sin A = \pm\sqrt{1 - \cos^2 A}$

$\tan A = \frac{1}{\cot A}$, $\tan A = \pm\sqrt{\sec^2 A - 1}$, $\tan A = \frac{\sqrt{1-\cos^2 A}}{\cos A}$

$\cos A = \frac{1}{\sec A}$, $\cos A = \pm\sqrt{1 - \sin^2 A}$, $\cos A = \frac{\sin A}{\tan A}$

To express all the functions of angle A in terms of sin A we have,

$\sin A = \sin A$; $\cos A = \sqrt{1 - \sin^2 A}$; $\tan A = \frac{\sin A}{\sqrt{1 - \sin^2 A}}$;

$\cot A = \frac{1}{\tan A} = \frac{\sqrt{1-\sin^2 A}}{\sin A}$; $\sec A = \frac{1}{\sqrt{1 - \sin^2 A}}$; $\csc A = \frac{1}{\sin A}$

In like manner the student should express all the functions in terms of cos A, tan A, cot A, sec A, csc A.

Equation and Identity. We know from algebra that an expression of equality which is true for one or more values of the unknown is called an equation. If the equality is true for all values of the unknown it is called an identity.

For example:

$$x^2 - 3x - 4 = (x - 4)(x + 1)$$

is an identity because both sides of the equation have the same value for all values of x; whereas

$$x^2 - 3x = 4$$

is true only for $x = 4$ and $x = -1$, and is not an identity.

The identity is an abstract formula true for all values of the unknown; while an equation always represents some concrete situation.

Both of these types arise in equations which involve trigonometric functions. The symbol $\equiv$ is sometimes used instead of $=$ to indicate identity.

Trigonometric Identities. The fundamental relations are identities because they are true for all values of the unknown. There are many other identities depending upon them. For they enable us to write an expression in many different forms. It is often important to be able to recognize that two expressions, although different in form, are really identical.

The truth of an identity can be established in many ways by means of the fundamental relations and algebraic principles. We shall illustrate only two of the most common methods:

(1) Reducing one member of the identity into the other;

(2) Reducing both sides of the identity to the same form.

Prove that, $\sin A \cdot \tan A = \dfrac{1}{\cos A} - \cos A$

$$\frac{1}{\cos A} - \cos A = \frac{1 - \cos^2 A}{\cos A}$$

$$= \frac{\sin^2 A}{\cos A}$$

$$= \sin A \cdot \frac{\sin A}{\cos A}$$

$$= \sin A \cdot \tan A$$

We have reduced the right hand member into the left hand member and thus verified the identity. We shall now reduce the left member into the right member:

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$$\sin A \cdot \tan A = \sin A \cdot \frac{\sin A}{\cos A}$$

$$= \frac{\sin^2 A}{\cos A}$$

$$= \frac{1 - \cos^2 A}{\cos A}$$

$$= \frac{1}{\cos A} - \frac{\cos^2 A}{\cos A}$$

$$= \frac{1}{\cos A} - \cos A$$

We shall reduce both members to the same form

$\sin A \cdot \tan A$	$\frac{1}{\cos A} = \cos A$
$= \sin A \quad \frac{\sin A}{\cos A}$	$= \frac{1 - \cos^2 A}{\cos A}$
$= \frac{\sin^2 A}{\cos A}$	$\frac{\sin^2 A}{\cos A}$

Because both members are equal to the same thing they are equal.

Note. In verifying identities, it is often advantageous to express each of the functions in terms of sines and cosines.

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Exercise.

Prove the following identities:

1. $\cos A \cdot \tan A = \sin A$.
2. $\cos^2 A - \sin^2 A = 2\cos^2 A - 1$.
3. $\cos^2 A - \sin^2 A = 2\sin^2 A$.
4. $\tan A + \cot A = \sec A \cdot \csc A$.
5. $\csc^2 A - \cot^2 A = 1$.
6. $\cos^2 A(1 + \tan^2 A) = 1$.
7. $\tan A = \sin A \cdot \sec A$.
8. $\sin^2 A(1 + \cot^2 A) = 1$.
9. $\cot^2 A - \cos^2 A = \cot^2 A \cdot \cos^2 A$.
10. $\cos A \cdot \tan A \cdot \csc A = 1$.
11. $\sec^2 A + \csc^2 A = \sec^2 A \cdot \csc^2 A$.
12. $\tan A \cdot \sin A + \cos A = \sec A$.
13. $\dfrac{\tan A}{\sqrt{1 + \tan^2 A}} = \sin A$.
14. $\dfrac{\cot A}{\sqrt{1 + \cot^2 A}} = \cos A$.
15. $\dfrac{\sin A}{\csc A} + \dfrac{\cos A}{\sec A} = 1$.
16. $\dfrac{1 + \tan^2 A}{1 + \cos^2 A} = \tan^2 A$.
17. $\cos^4 A - \sin^4 A = \cos^2 A - \sin^2 A$.
18. $\dfrac{\csc A}{\tan A + \cot A} = \cos A$.
19. $\dfrac{1}{\sin A} - \sin A = \dfrac{\cos A}{\tan A}$.
20. $\dfrac{\cos A + \sin A}{1 + \tan A} = \cos A$.
21. $\dfrac{1}{\tan^2 A} = \dfrac{1}{\sin^2 A} - 1$.
22. $\sqrt{1 - \sin^2 A} = \sin A \cdot \dfrac{1}{\tan A}$.
23. $\dfrac{\sin A}{1 + \cos A} = \dfrac{1 - \cos A}{\sin A}$.
24. $\dfrac{\tan A}{\sec A - 1} = \dfrac{\sec A + 1}{\tan A}$.
25. $\dfrac{\tan A - 1}{\tan A + 1} = \dfrac{1 - \cot A}{1 + \cot A}$.
26. $\sec^4 A - \tan^4 A = 1 + 2\tan^2 A$.
27. $\sec^2 A + \csc^2 A = \sec^2 A \cdot \csc^2 A$.
28. $\sec A - \tan A \cdot \sin A = \cos A$.

Inverse Trigonometric Functions. The student is familiar with inverse relations such as addition and subtraction, multiplication and division, raising to a power and taking a root. For example, $3 + 4 = 7$ and $4 + 3 = 7$; $\dfrac{6}{2} = 3$ and $3 \cdot 2 = 6$; $y = x^2$ and $x = \pm\sqrt{y}$.

In like manner, if we have

$$x = \sin A$$

we can express this relation by saying

"A is the angle whose sine is equal to x".

This statement is written, $A = \sin^{-1} x$, or $A = \text{arc sin } x$.

In the same way, $A = \cos^{-1} y$, or $A = \text{arc cos } y$, means "A is the angle whose cosine is y".

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Observe that $A = \tan^{-1} x$, and $x = \tan A$, are merely different forms of expressing the same relation between A and x. In the first, A is a function of x; in the second x is a function of A. Hence, they are inverse functions.

The student should not confuse the symbol of the inverse functions with the negative exponent. When the exponent of a function is negative one, it should be written $(\sin A)^{-1}$ and not $\sin^{-1} A$. All other exponents are written without a parenthesis as

$\sin^2 A$ for $(\sin A)^2$ and $\cos^{-2} x$ for $(\cos A)^{-2}$.

A given angle has only one sine, one cosine, one tangent, etc. But when the inverse function is given there is an unlimited number of values for the angle. To illustrate, $\sin 30^\circ = \frac{1}{2}$ and no other value. But $\sin^{-1} \frac{1}{2}$ may be 30°, 150°, 390°, 510°, -210°, -330°, or any angle differing from the two smallest positive angles by a multiple of 360°.

The general angle may be written:

$$\sin^{-1}\tfrac{1}{2} = (n \cdot 360^\circ + 30^\circ) \text{ and } (n \cdot 360^\circ + 150^\circ)$$

where n is a whole number positive or negative.

The smallest positive angle is called the Principal Angle and is sometimes indicated by a capital letter thus,

$$\text{Arc } \sin \tfrac{1}{2} = \text{Sin}^{-1} \tfrac{1}{2} = 30^\circ \text{ and no other value.}$$

Remember that the inverse functions are angles.

Exercise.

Construct the following angles, find their functions, and the values of the angles.

1.. $\sin^{-1} \frac{\sqrt{3}}{2}$.

2.. $\cos^{-1}(-\frac{1}{2})$.

3.. arc tan (-1).

4.. arc sec 2.

5.. $\cot^{-1} \frac{1}{\sqrt{3}}$.

6.. $\sin^{-1}(-1)$,

Find the value of:

7.. $\tan (\sin^{-1} \frac{1}{2})$.

8.. $\sin (\cos^{-1} v)$.

9.. $\cos (\tan^{-1} v)$.

10. $\tan^{-1} (\sin 90^\circ)$.

11. $\sin (\sin^{-1} v)$.

12. $\cos^{-1} (\cos B)$.

13. $\tan^{-1} (\tan 30^\circ)$.

14. $\sin \left(\sin^{-1} \frac{1}{2} + \cos^{-1} \frac{1}{\sqrt{2}} \right)$

Trigonometric Equations. In trigonometry, we often need to find the value of the angle which satisfies an equation which contains some trigonometric function. The method of solving these trigonometric equations is similar to that of solving algebraic equations since algebraic principles apply to functions as they do to letters.

Example 1, Solve, $\cos^2 A = \frac{1}{4}$

Take the square root of both sides,

$$\cos A = \tfrac{1}{2} \text{ and } \cos A = -\tfrac{1}{2}.$$

Hence, $A = \cos^{-1}\tfrac{1}{2}$ or $A = \cos^{-1}(-\tfrac{1}{2})$.

. . $A = 60^\circ$ and 300° or $A = 120^\circ, 240^\circ$.

If we substitute these values in the given equation, the equation is satisfied. Hence they are solutions.

The general angle including all the solutions may be written

$$A = (n \cdot 360^\circ + 60^\circ) \text{ and } A = (n \cdot 360^\circ + 120^\circ).$$

Example 2. $\cos^2 A - \sin^2 A = \sin A$

$$1 - 2\sin^2 A - \sin A = 0.$$

or, $$2\sin^2 A + \sin A - 1 = 0$$

This is a quadratic equation which can be solved by factoring,

$$(2\sin A - 1)(\sin A + 1) = 0.$$

This product is 0, if

$2 \sin A - 1 = 0$ or $\sin A + 1 = 0$

$\sin A = \frac{1}{2}$ $\sin A = -1$

$A = \sin^{-1}\frac{1}{2}$ $A = \sin^{-1}(-1)$

$A = 30^\circ, 150^\circ$ $A = 270^\circ$

These three values satisfy the given equation and are, therefore, solutions.

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Example 3. Solve $\cos A - \sin A = 1$.

By substitution, $\cos A - \sqrt{1 - \cos^2 A} = 1$.

or $\sqrt{1 - \cos^2 A} = \cos A - 1$

Squaring, $1 - \cos^2 A = \cos^2 A - 2 \cos A + 1$

or $2 \cos^2 A - 2 \cos A = 0$

Factoring $\cos A(\cos A - 1) = 0$

$\cos A = 0$ $\qquad$ $\cos A = 1$

$A = 90^\circ$ or 270° $\qquad$ $A = 0^\circ$

Testing these values in the given equation, we find that 0° and 270° satisfy the equation, while 90° does not. Hence 90° is not a solution.

The general solutions are $A = n \cdot 360^\circ$ or $n \cdot 360^\circ + 270^\circ$.

Suggestions.

1. Simplify the equation by means of the fundamental relations.

2. Reduce the functions into one function.

3. If the equation is quadratic in form solve by factoring, completing the square, or by formula.

4. Guard against division by a variable which may result in losing a solution.

5. Every result must be checked to determine whether it is a solution or not. Substitution in the original equation helps to detect extraneous roots.

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Exercise.

Find all the angles less than 360^{o} which satisfy the following equations:

1. $2 \sin^2 A = 1.$
2. $3 \tan^2 A = 4.$
3. $4 \cos^2 A = 3.$
4. $2 \tan A \cos A = 1.$
5. $\tan A + \cot A = 2.$
6. $\csc^2 A = 1.$
7. $3 \sec^2 A = 4.$
8. $2 \tan^2 A + 1 = 0$
9. $\sin A = .66153.$
10. $\tan A = - .45047.$
11. $\cos A = .45269.$
12. $\log \sin A = 9.73805 - 10.$
13. $\sec A = 1.30541.$
14. $\sin A - \cos A = 0.$
15. $2 \cos^2 A + 7 \sin A = 5.$
16. $2 \sin^2 A - 3 \sin A + 1 = 0.$
17. $2 \sin A + \cos A = 2.$
18. $1 - 4 \cos^2 A + \tan^2 A = 1.$
19. $\sin A \cos A = \pm \frac{\sqrt{3}}{4}$
20. $\cos A = \sin A - 1.$
21. $4 \sec A - 1 = 8 \tan A.$
22. $\cot A - \cos A = \sin A + \tan A.$
23. $\tan A + \cot A = 2 \sec A.$
24. $\cos A - \cos^2 A + \sin^2 A = 1.$

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Oblique Triangles.

In the preceding sections we learned to solve right triangles. There are, however, oblique triangles which need to be solved. The solution of such triangles is the aim of this section.

An oblique triangle is one which does not have a right angle. As stated before, when any three independent parts of an oblique triangle are given, the triangle is determined and can be constructed and solved.

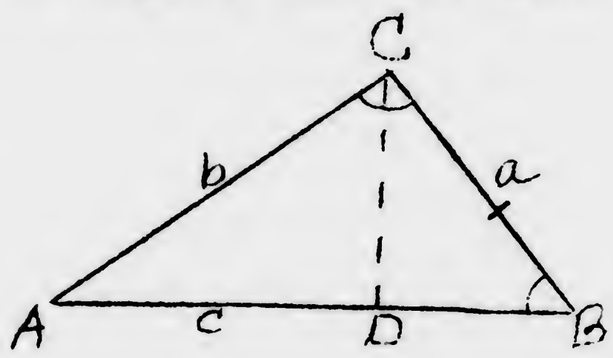
There are four possible cases.

(1) Given two angles and one side.
(2) Given two sides and the angle opposite one of them.
(3) Given two sides and the included angle.
(4) Given the three sides.

Each one of these cases can be solved by the method of solving the right triangle. It is only necessary to draw a convenient altitude and solve the resulting right triangles.

Case I. When two angles and one side are given.

Given, $a = 24.5$
$B = 48^\circ\ 20'$
$C = 67^\circ\ 5'$

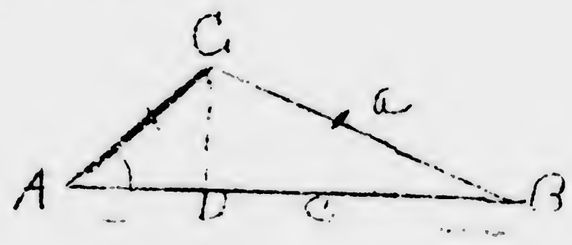


Find
$A =$
$b =$
$c =$

Draw the altitude CD to AB. Solve the right triangle B D C. Then, knowing C D and angle A C D, solve the right triangle A D C. Complete the solution of this example.

Case II. When two sides and the angle opposite one of them are given.

Given, $a = 70$
$b = 50$
$A = 65^\circ$

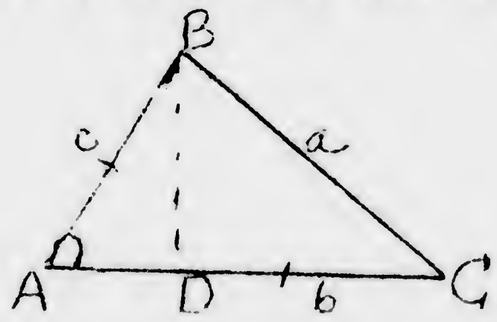


Find
$B =$
$C =$
$c =$

Draw the altitude CD. Solve the right triangle A D C and then triangle B D C. Complete the solution of this porblem

Case III. When two sides and the included angle are given.

Given, $b = 4.5$
$c = 6.3$
$A = 75^\circ\ 20'$



Find
$a =$
$B =$
$C =$

Draw altitude BD. Solve triangle A D B and then solve triangle C D B. Complete the solution.

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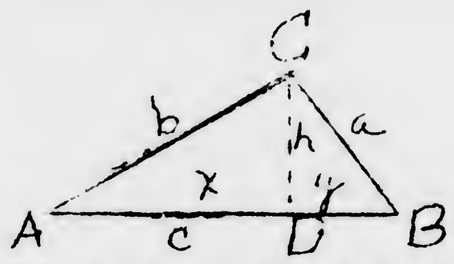
Note: In cases I, II, and III, when one of the given parts is an angle, care must be taken to draw the altitude so that one of the resulting triangles will contain two of the given parts.

Case IV. When the three sides are given.

Given, a = 40
b = 30
c = 55



Find
A =
B =
C =

Draw altitude CD and call it h. From the triangle A D C, $h^2 = b^2 - x^2$. From the triangle B D C, $h^2 = a^2 - y^2 = a^2 - (C - x)^2$

Therefore, $b^2 - x^2 = a^2 - (c - x)^2$

or, $b^2 - x^2 = a^2 - c^2 + 2cx - x^2$

Hence, $x = \frac{b^2 + c^2 - a^2}{2c}$

After finding x and y the two right triangles can be solved. Complete the solution of this example.

Exercise.

Solve the following oblique triangles by the method of the right triangle:

1. a = 24.63
   A = 82° 42′
   B = 37° 15′

2. a = 4.935
   c = 8.367
   A = 27° 40′

3. a = .84
   b = .91
   c = 1.16

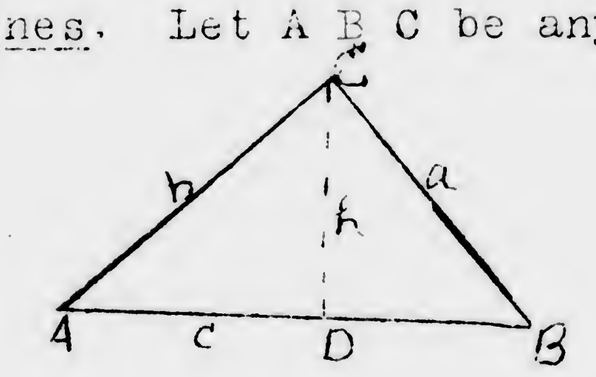
4. a = 52.4
   b = 75.1
   c = 43.8

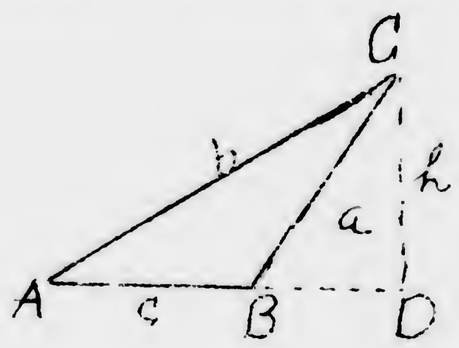
5. b = 5.203
   c = 6.075
   A = 65° 30′ 30″

6. a = 72
   c = 43
   B = 125°

In order to avoid the labor of dividing oblique triangles into right triangles, we shall now derive some formulas which will help us to solve oblique triangles more directly.

The Law of Sines. Let A B C be any triangle. Draw C D perpendicular to A B.





In each figure, $\sin A = \frac{h}{b}$

and $\sin B = \frac{h}{a}$

Dividing equals by equals we get

$$\frac{\sin A}{\sin B} = \frac{h}{b} \div \frac{h}{a} = \frac{h}{b} \times \frac{a}{h}$$

$$\therefore \quad \frac{\sin A}{\sin B} = \frac{a}{b}$$

This may be written, $\frac{a}{\sin A} = \frac{b}{\sin B}$

In like manner by drawing an altitude from C to A B,

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

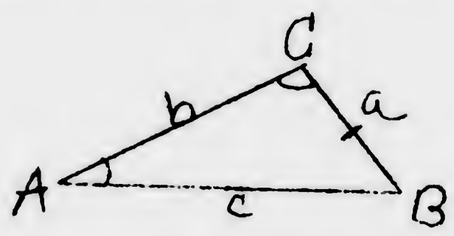
That is, in any triangle, the three sides are proportional to the sines of the opposite angles. This is the Law of Sines.

It is evident that each equation of the law of sines involves four elements of a triangle. If any three are given, the fourth can be found.

The law of sines is used to solve triangles which fall under Cases I and II.

Example for Case I. When two angles and one side are given.

Given, $a = 2.024$
$C = 103^\circ\ 36'$
$A = 19^\circ\ 21'$



Find
$B = 57^\circ\ 3'$
$b = 5.126$
$c = 5.9373$

Graphic solution: $b = 5$, $c = 6$

Formulas: $B = 180 - (A + C) = 180^\circ - 122^\circ\ 57' = 57^\circ\ 3'$

To find c,

$$\frac{c}{a} = \frac{\sin C}{\sin A}$$

$$c = \frac{a \sin C}{\sin A}$$

$\log c = \log a + \log \sin C - \log \sin A$
$\log 2.024 = 0.30621$
$\log \sin 103^\circ\ 36' = 9.98765 - 10$
$10.29386 - 10$
$\log \sin 19^\circ\ 21' = 9.52027 - 10$
$\log c = .77359$
$c = 5.9373$

To find b,

$$\frac{b}{a} = \frac{\sin B}{\sin A}$$

$$b = \frac{a \sin B}{\sin A}$$

$\log b = \log a + \log \sin B - \log \sin A$
$\log 2.024 = 0.30621$
$\log \sin 57^\circ\ 3' = 9.92384 - 10$
$10.23005 - 10$
$\log \sin 19^\circ\ 21' = 9.52027 - 10$
$\log b = 0.70978$
$b = 5.126$

Remarks: To find the sine of an obtuse angle A in the table, find the sine of its supplement, or $(180^\circ - A)$.
In selecting the formulas take the part required and three parts that are known (preferably the given parts). Begin the formula with the required part.

Checking. In practical work a problem is not considered solved until it has been checked. The graphic solution by a scale drawing is a great aid in detecting large errors. But for accurate work a mathematical check should be used. In general, a solution of the problem by some different method is used as a check. In working a check the logarithms must be looked up in the table and not taken from the first solution.

The most reliable and convenient check may be obtained by one of the Mollweide's equations, for each of them contains all of the six parts of a triangle. One of these formulas is,

$$\frac{a-b}{c} = \frac{\sin \frac{1}{2}(A - B)}{\cos \frac{1}{2}C}, \quad \text{or} \quad \frac{b-a}{c} = \frac{\sin \frac{1}{2}(B - A)}{\cos \frac{1}{2}C}$$

Let us apply it to the given example.
Log (b - a) = log c + log sin $\frac{1}{2}$(B-A) - log cos $\frac{1}{2}$C.

Check.

$b-a = 3.102$
$B-A = 37^\circ\ 42'$
$\frac{1}{2}(B-A) = 18^\circ\ 51'$
$\frac{1}{2}C = 51^\circ\ 48'$

$\log c = .77359$
$\log \sin \frac{1}{2}(B-A) = 9.50933 - 10$
$= 10.28292 - 10$
$\log \cos \frac{1}{2}C = 9.79128 - 10$
$\log(b-a) = .49164$
$b-a = 3.102$

Checks.

Exercise.

Solve the following triangles:

1. $a = 53.68$, $A = 71^\circ 10'$, $B = 40^\circ 25'$

2. $b = .705$, $B = 95^\circ 40'$, $C = 18^\circ 45'$

3. $c = 325.85$, $A = 110^\circ\ 42'.6$, $B = 41^\circ\ 26'$

4. $b = 30.08$, $A = 37^\circ\ 12'$, $C = 52^\circ 40' 20''$

5. $c = 8.309$, $A = 16^\circ 34'$, $C = 64^\circ$

6. $a = 452.00$, $A = 57^\circ 24'$, $B = 102^\circ\ 3'$

7. $c = 764.3$, $C = 100^\circ\ 50'$, $A = 12^\circ\ 16'$

8. $a = .086$, $B = 43^\circ\ 35'$, $C = 17^\circ 5' 40''$

9. $b = .175$, $A = 25^\circ\ 25'\ 25''$, $B = 62^\circ\ 28'\ 10''$

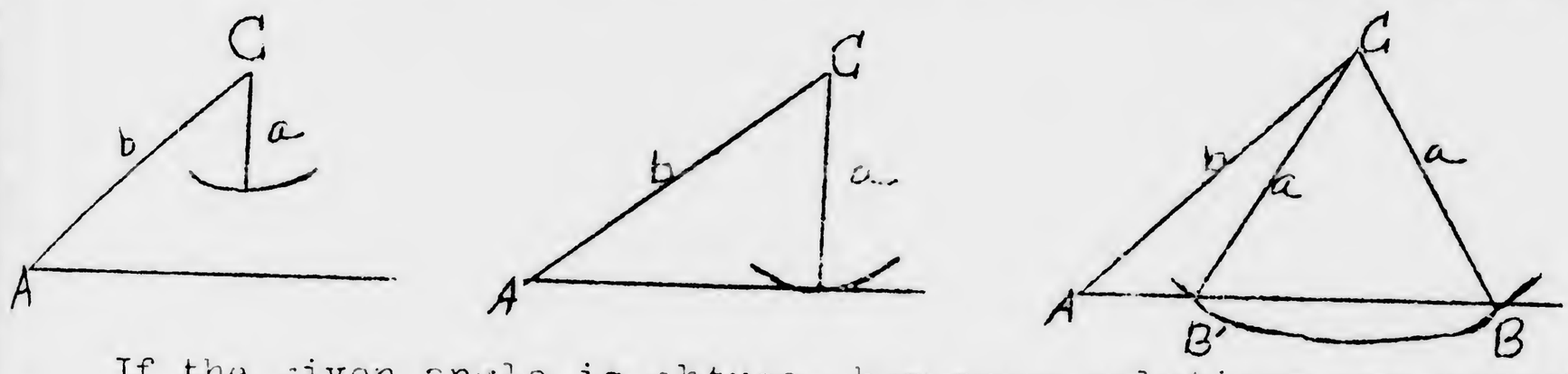
Solutions of Case II. When two sides and the angle opposite one of them are given. From the given parts it is seen that the law of sines helps us to solve the triangle by solving for the angle opposite the other given side.

Solving for an Angle. If we find from a triangle that the cosine of an angle is positive, this means that the angle is acute. If we find that the cosine is negative, it means that the angle is obtuse. (Why?) To illustrate.

If cos A = .6820, then A is 47°. If cos B = - .5736, then B is obtuse, and its supplement has its cosine equal to .5736. By the table the supplement of B is 55°; hence B = 125°.

If we find from a triangle that the sine of an angle is positive, the angle may be acute, or may be obtuse. (Why?) Can the sine of the angle of a triangle have a negative value? Why? If sin A = .5736, it means that A = 35° by the tables, or A - 145°(180° - 35°), since the sine of an angle is equal to the sine of its supplement. This situation may lead to two solutions. This is known as the "ambiguous case".

Whether there is one solution, two solutions, or no solution can readily be determined by making a careful construction of the triangle from the given parts. Let A, a, b, be the given parts.



If the given angle is obtuse, how many solutions are possible? Why?

If the side opposite the given angle is larger than the other given side, how many solutions are possible?

The possibilities of this case would be discovered in the process of solving if the student would remember the following facts:

1. If, on computing the sine of an angle, we find its value to be greater than unity(or the value of its logarithm to be greater than zero) the triangle is impossible.

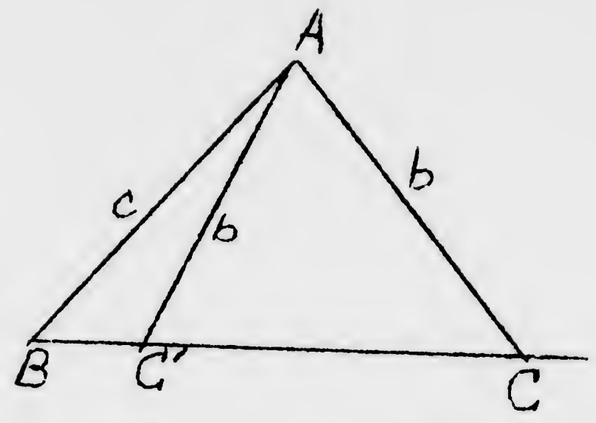
2. If the sine of an angle is a positive proper fraction, there are <u>two</u> possibilities for the angle.

3. The sum of any two angles of a triangle must be less than 180 degrees.

4. The greater angle is opposite the greater side.

Example.

Given
b = 345
c = 534
B = 25° 25′



Find
C = 41° 37′ C′ = 138° 23′
A = 112° 58′ A′ = 16° 12′
a = 740.1 a′ = 224.3

Construction. Construct the given angle B and on one of its sides lay off B A = c. With A as a center and a radius equal to b describe an arc of a circle to determine the vertex of the third angle. Since the arc cuts the side B C in two points C and C′, two triangles are possible A B C and A B C′.

From the drawing,

C = 40° and C′ = 138°
A = 112° A′ = 16°
a = 740 a′ = 220

Formulas:

To find C, $\frac{\sin C}{\sin B} = \frac{c}{b}$

$\sin C = \frac{c \sin B}{b}$

log sinC = log c + log sin B - log b
log 534 = 2.7275
log sin 25° 25′ = 9.6326-10
12.3601-10
log 345 = 2.5378
log sinC = 9.8223-10
C = 41° 37′ or
= 138° 23′
B+C = 67° 2′ or
= 163° 48′
A = 112° 58′
16° 12′

To find a, $\frac{a}{b} = \frac{\sin A}{\sin B}$

$a = \frac{b \sin A}{\sin B}$

loga = logb + logsinA - logsinB
log 345 = 2.5378
log sin 112°58′ = 99.9641-10
12.5019-10
log sin 25° 25′ = 9.6326-10
log a = 2.8693
a = 740.1

log b = 2.5378
logsin 16°12′ = 9.4456 - 10
11.9834 - 10
log sin 25° 25′ = 9.6326 - 10
log a′ = 2.3508
a′ = 224.3

Check both solutions by Mollweide's equation.

Exercise.

Find out whether the triangles corresponding to the following data are possible, how many solutions there are, and the values of the unknown parts.

1. a = 45.6, b = 72.3, B = 106° 20′

2. b = 6.09, c = 4.30, B = 52° 40′

3. a = 2.35, b = 1.12, B = 68° 45′

4. b = 30.75, c = 22.06, C = 35° 10′

5. a = .752, c = .905, A = 28° 40′ 20″

6. a = .356, b = .4072, B = 73° 5′

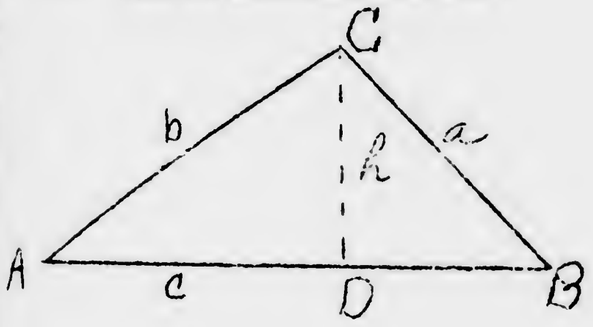
7. a = 126.4, b = 182.8, A = 78° 50′

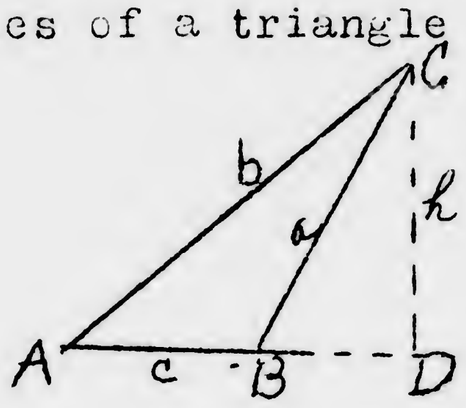
8. b = .1763, c = .0905, C = 95° 10′

9. a = 5280, c = 4020, A = 42° 40′

The Law of Cosines.

One of the most useful properties of a triangle is known as the Law of Cosines.





$a^2 = h^2 + \overline{DB}^2$
$= h^2 + (c - AD)^2$
$= h^2 + c^2 - 2C.\overline{AD} + \overline{AD}^2$
$= b^2 + c^2 - 2C.\overline{AD}$
(1) $a^2 = b^2 + c^2 - 2bc \cos A$

$a^2 = h^2 + \overline{BD}^2$
$= h^2 + (\overline{AD} - C)^2$
$= h^2 + \overline{AD}^2 - 2C.AD + C^2$
$= b^2 + c^2 - 2C.AD$
$a^2 = b^2 + c^2 - 2bc.\cos A$

In like manner we can prove,

(2) $b^2 = a^2 + c^2 - 2ac \cos B$
(3) $c^2 = a^2 + b^2 - 2ab \cos C$

These formulas may be stated as follows:

> The square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of these two sides and the cosine of their included angle. This is the Law of Cosines. This law should be carefully memorized. Observe that every formula begins and ends with the same letter.

Case III. It is obvious that formulas (1), (2) and (3) enable us to solve triangles classified under Case III, where two sides and their included angle are given.

Case IV. If we solve formulas '1), (2) and (3) for cos A, cos B and cos C respectively, we have:

$$(4)\quad \cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$(5)\quad \cos B = \frac{a^2+c^2-b^2}{2ac}$$

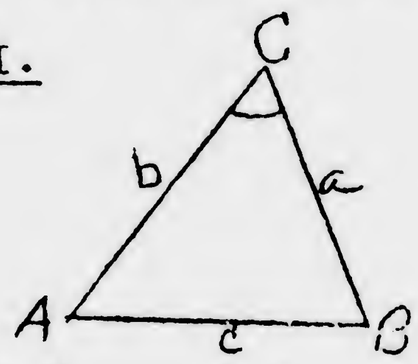
$$(6)\quad \cos C = \frac{a^2+b^2-c^2}{2ab}$$

These three formulas enable us to solve the triangles of Case IV, when the three sides are given.

Thus by using the Law of Cosines, we can solve oblique triangles classified under Cases III, and IV.

Notice that logarithms cannot be used conveniently with formulas (1) to (6). Nevertheless the Law of Cosines commonly used in practical work, especially in engineering practice and mechanics, the calculations being made not with logarithms but with tables of squares or with the slide rule.

Example for Case III.



Given
a = 8
b = 10
C = 45°

Find
c = 7.13
A =
B=

Graphic solution c = 7, A = , B =

Formulas:

For c,

$$c^2 = a^2+b^2 - 2ac\cos C$$

$$c = \sqrt{64+100 - 2\cdot 8\cdot 10\cdot .707}$$

$$c = \sqrt{50.88} = 7.133$$

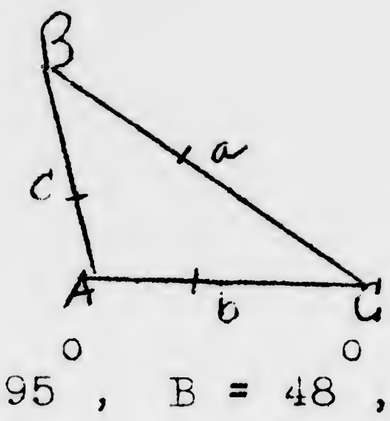
To find A, $\frac{\sin A}{\sin C} = \frac{a}{c}$

To find B, $\frac{\sin B}{\sin C} = \frac{b}{c}$

Complete the solutions and check by Mollweide's equation.

Example for Case IV.

Given
a = 20
b = 15
c = 12



Find
A = 94° 56′
B = 48° 20′
C = 36° 43′
A+B+C = 179° 59′

Graphic solution A = 95°, B = 48°, C = 36°.

Formulas:

$$\text{Cos } A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \text{Cos } B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\text{Cos } A = \frac{225 + 144 - 400}{2 \times 15 \times 12} \qquad = \frac{400 + 144 - 225}{480}$$

$$= \frac{-31}{350} \qquad = \frac{319}{480}$$

Cos A = - .0861
A = (180° - 85° 4′)
A = 94° 56′

Cos B = .6648
B = 48° 20′

$$\text{Cos } C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{400 + 225 - 144}{600}$$

$$= \frac{481}{600}$$

Cos C = .8016
C = 36° 43′

Observe that in Case IV, A + B + C = 180° is a convenient check if the three angles are solved independently.

Exercise.

When should the cosine law be used to solve a triangle? The sine law?

Solve the following triangles and check:

1. a = 24, b = 30, C = 53° 20′

2. a = 16, b = 25, c = 30

3. b = 120, c = 95, A = 110° 26′

4. a = 27.5
   b = 35.2
   c = 40.6

5. a = 16.75
   c = 9.48
   B = 30°

6. a = 2.735
   b = 5.064
   C = 6.803

7. a = .785
   b = .632
   c = 1.105

8. a = 75
   b = 60
   C = 120°

9. b = 15.08
   c = 20.07
   A = 135°

10. a = 5280
    b = 4000
    C = 75° 30′

11. a = 1760
    b = 2050
    c = 2500

12. a = .941
    c = 1.254
    B = 65° 30′ 20″

13. b = .472
    c = .324
    A = 78° 40′

14. a = 1.436
    c = 1.141
    B = 42° 14′ 35″

15. a = .054
    b = .086
    c = .101

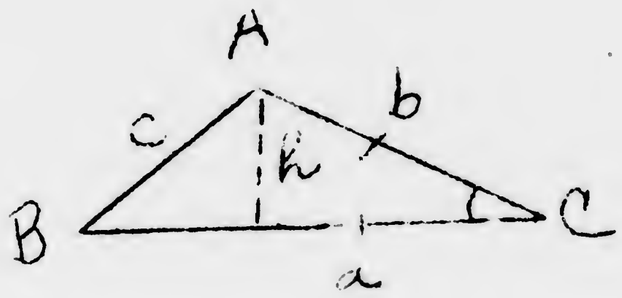
The Area of Oblique Triangles.

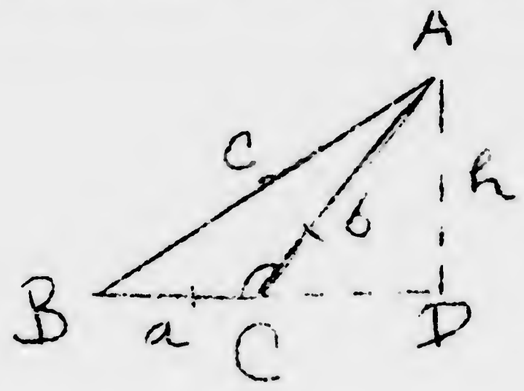
The area of any triangle is equal to half the base times the altitude.

$$K = \tfrac{1}{2} ah$$

This formula applies to right triangles and oblique triangles alike and is the basis from which special formulas are derived. To find the area of a triangle, therefore, we may first find the length of a convenient altitude and then multiply by half the corresponding base. This is the best procedure in Cases I. and II.

Case III. For Case III, when two sides and the included angle are given, there is a special formula which shortens the work. Let the sides a, b of a triangle and the included angle C be given.





Drop a perpendicular from A to BC and call it h.

Then, $K = \tfrac{1}{2} ah$

Now h can be expressed in terms of the given quantities.

For $\sin C = \frac{h}{b}$

$$\therefore \quad h = b \sin C$$

Hence $K = \tfrac{1}{2} ab \sin C$, or $2K = ab \sin C$

Similarly $K = \tfrac{1}{2} bc \sin A = \tfrac{1}{2} ac \sin B$.

That is, the area of a triangle is equal to half the product of any two sides times the sine of their included angle.

Case IV. There is also a convenient formula for the area when the three sides are given. To derive this formula, we start with the formula already developed for Case III. i. e., $K = \tfrac{1}{2} ab \sin C$, and make use of the Law of Cosines and the relation $\sin^2 A + \cos^2 A = 1$ as follows:

$$K = \tfrac{1}{2} ab \sin C$$

$$K^2 = \tfrac{1}{4} a^2 b^2 \sin^2 C = \tfrac{1}{4} a^2 b^2 (1 - \cos^2 C)$$

$$= \tfrac{1}{4} a^2 b^2 (1 + \cos C)(1 - \cos C)$$

By the Cosine Law,

$$= \frac{a^2 b^2}{4}\left(1 + \frac{a^2 + b^2 - c^2}{2\,ab}\right)\left(1 - \frac{a^2 + b^2 - c^2}{2\,ab}\right)$$

$$= \frac{a^2 b^2}{4} \cdot \frac{2\,ab + a^2 + b^2 - c^2}{2\,ab} \cdot \frac{2\,ab - a^2 - b^2 + c^2}{2\,ab}$$

$$= \frac{1}{16}\left[(a+b)^2 - c^2\right]\left[c^2 - (a-b)^2\right]$$

$$= \frac{1}{16}(a+b+c)(a+b-c)(c+a-b)(c-a+b)$$

This expression becomes simple if we denote half the perimeter by s, so that the four factors become

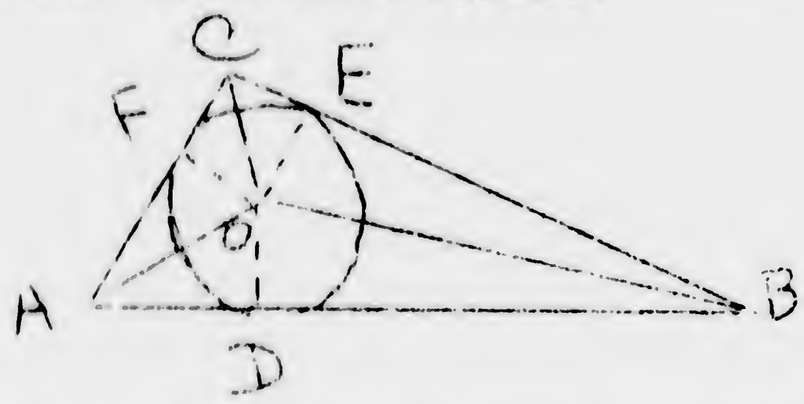
$$a + b + c = 2s$$
$$a + b - c = 2s - 2c = 2(s - c)$$
$$c + a - b = 2s - 2b = 2(s - b)$$
$$c - a + b = 2s - 2a = 2(s - a)$$

and so $$K^2 = s(s - a)(s - b)(s - c)$$

Hence (1) $$K = \sqrt{s(s - a)(s - b)(s - c)}$$

well known equation known in geometry as Hero's Formula.

A special form of this equation is particularly interesting. In the given triangle whose sides are a, b, c, let o be the center and r the radius of the inscribed circle.



Area ABC = area OAB + area OBC + area OCA, or

$$K = \tfrac{1}{2} ar + \tfrac{1}{2} br + \tfrac{1}{2} cr$$
$$= \tfrac{1}{2}(a + b + c)\, r$$

(2) $K = rs$

From (1) and (2) we have

$$rs = \sqrt{s(s - a)(s - b)(s - c)}$$

Hence, $$r = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}}$$

This formula enables us to compute the radius of the inscribed circle when the three sides are given.

4. a = 27.5
b = 35.2
c = 40.6

5. a = 16.75
c = 9.48
B = 30°

6. a = 2.735
b = 5.064
C = 6.803

7. a = .785
b = .632
c = 1.105

8. a = 75
b = 60
C = 120°

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c = 20.07
A = 135°

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C = 75° 30′

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TIGHT BINDING

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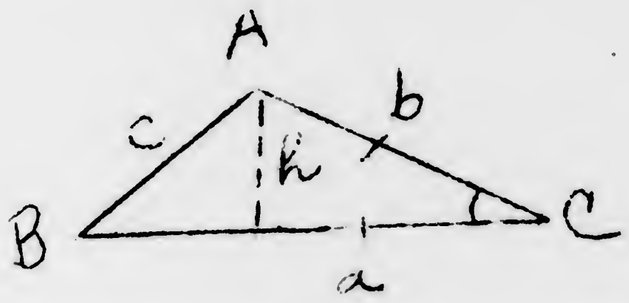

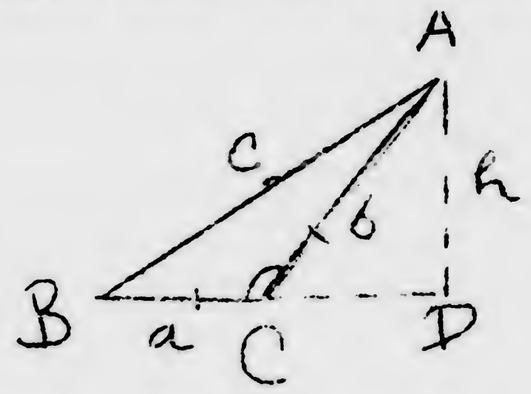

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For $\sin C = \frac{h}{b}$

$$\therefore \quad h = b \sin C$$

Hence $K = \tfrac{1}{2} ab \sin C$, or $2K = ab \sin C$

Similarly $K = \tfrac{1}{2} bc \sin A = \tfrac{1}{2} ac \sin B$.

That is, the area of a triangle is equal to half the product of any two sides times the sine of their included angle.

Case IV. There is also a convenient formula for the area when the three sides are given. To derive this formula, we start with the formula already developed for Case III. i. e., $K = \tfrac{1}{2} ab \sin C$, and make use of the Law of Cosines and the relation $\sin^2 A + \cos^2 A = 1$ as follows:

$$K = \tfrac{1}{2} ab \sin C$$

$$K^2 = \tfrac{1}{4} a^2 b^2 \sin^2 C = \tfrac{1}{4} a^2 b^2 (1 - \cos^2 C)$$

$$= \tfrac{1}{4} a^2 b^2 (1 + \cos C)(1 - \cos C)$$

By the Cosine Law,

$$= \frac{a^2 b^2}{4}\left(1 + \frac{a^2 + b^2 - c^2}{2\,ab}\right)\left(1 - \frac{a^2 + b^2 - c^2}{2\,ab}\right)$$

$$= \frac{a^2 b^2}{4} \cdot \frac{2\,ab + a^2 + b^2 - c^2}{2\,ab} \cdot \frac{2\,ab - a^2 - b^2 + c^2}{2\,ab}$$

$$= \frac{1}{16}\left[(a+b)^2 - c^2\right]\left[c^2 - (a-b)^2\right]$$

$$= \frac{1}{16}(a+b+c)(a+b-c)(c+a-b)(c-a+b)$$

TIGHT BINDING

This expression becomes simple if we denote half the perimeter by s, so that the four factors become

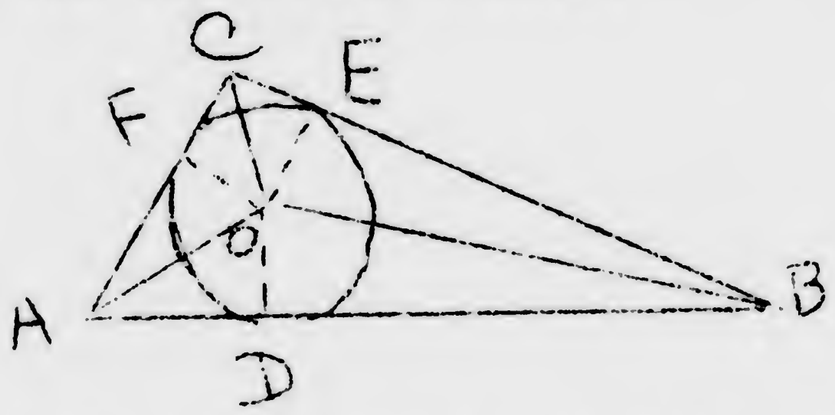
$$a + b + c = 2s$$
$$a + b - c = 2s - 2c = 2(s - c)$$
$$c + a - b = 2s - 2b = 2(s - b)$$
$$c - a + b = 2s - 2a = 2(s - a)$$

and so $$K^2 = s(s - a)(s - b)(s - c)$$

Hence (1) $$K = \sqrt{s(s - a)(s - b)(s - c)}.$$

well known equation known in geometry as Hero's Formula.

A special form of this equation is particularly interesting. n the given triangle whose sides are a, b, c, let o be the center nd r the radius of the inscribed circle.



Area ABC = area OAB + area OBC + area OCA, or

$$K = \tfrac{1}{2} ar + \tfrac{1}{2} br + \tfrac{1}{2} cr$$
$$= \tfrac{1}{2}(a + b + c)\ r$$

(2) $$K = rs$$

From (1) and (2) we have

$$rs = \sqrt{s(s - a)(s - b)(s - c)}$$

Hence, $$r = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}}$$

This formula enables us to compute the radius of the inscribed ircle when the three sides are given.

TIGHT BINDING
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Example.

Given,

a = 27.5
b = 30.4
c = 41.3
2s = 99.2
s = 49.6
s - a = 22.1
s - b = 19.2
s - c = 08.3
(Check) 2s = 99.2

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\log K = \tfrac{1}{2}\left[\log s + \log(s-a) + \log(s-b) + \log(s-c)\right]$$

log 49.6 =
log 22.1 =
log 19.2 =
log 08.3 =
log K = ½( )

log K =

K =

Complete the computations of this example.

Exercises.

Solve the following triangles and find their areas:

1. a = 16, b = 24, C = 45°

2. b = 5.2, c = 4.5, A = 30°

3. a = 62, c = 75, B = 60°

4. a = 25, b = 35, c = 45°

5. a = 128, b = 106, C = 40° 20′

6. a = .7306, b = .9075, c = 1.2563

7. a = 31.5, c = 40.2, B = 75° 15′

8. a = 1760, b = 2053, c = 3000

9. b = 519, c = 582, A = 115°

10. a = 60.6, B = 48° 20′, C = 57° 40′

11. a = 7.34, b = 9.05, B = 71° 25′

12. b = 23.05, c = 20.68, C = 42° 55′

13. a = 250, A = 72° 4′, C = 41° 57′ 36″

14. b = 123, A = 29° 17′, C = 135°

15. A = 45.49, c = 41.02, B = 62° 9′ 38″

TIGHT BINDING

16. For additional practice, find the areas in other exercises on the solution of triangles.

17. Find the radii of the inscribed circles in exercises 4, 6, and 8.

18. Find the areas of the inscribed circles in exercises 4, 6 and 8.

19. Two sides of a triangle are 23.18 and 18.43 and they include a right angle. Find the area.

20. Two sides and the included angle of a triangle are 1250 and 1560 and 35°; and two sides the included angle of another triangle are 1250 and 1560 and 145°. How do their areas compare?

21. Two sides of a parallelogram are 92.5 and 80.4 and their included angle is 105°. Find the area. Find the lengths of the diagonals.

22. Show that the area of a quadrilateral is equal to one half the product of its diagonals times the sine of the included angle.

23. The diagonals of a quadrilateral are 54 ft. and 68 ft. They intersect at an angle of $57^\circ 40'$; find the area.

24. $A = 43^\circ 20'$, $B = 61^\circ 5'$, $s = 58.6$. Find the area and the sides of the triangle.

25. $K = 560$, $a = 25$, $b = 46.3$; solve the triangle.

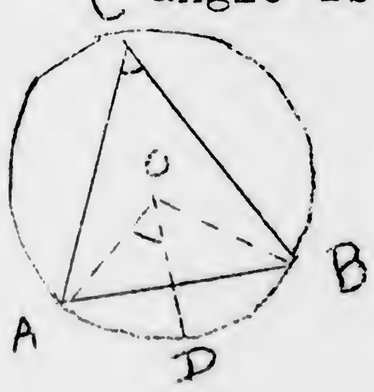
26. There is a piece of land with sides 56.3 rods, 59.1 rods, and 70.5 rods. Find the area in rods; in acres.

27. Given $a = 50$, $C = 65^\circ 40'$, area = 124; find the other parts of the triangle.

28. One side of a piece of land is 568 ft. and bears N. $81^\circ 40'$ E; another side is 609 ft. and bears S $15^\circ 30'$ W. Find the area.

29. Find the area of an equilateral triangle inscribed in a circle whose radius is 15.6 ft.

30. Show that twice the radius of a circle circumscribing a triangle is given by the relation



$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

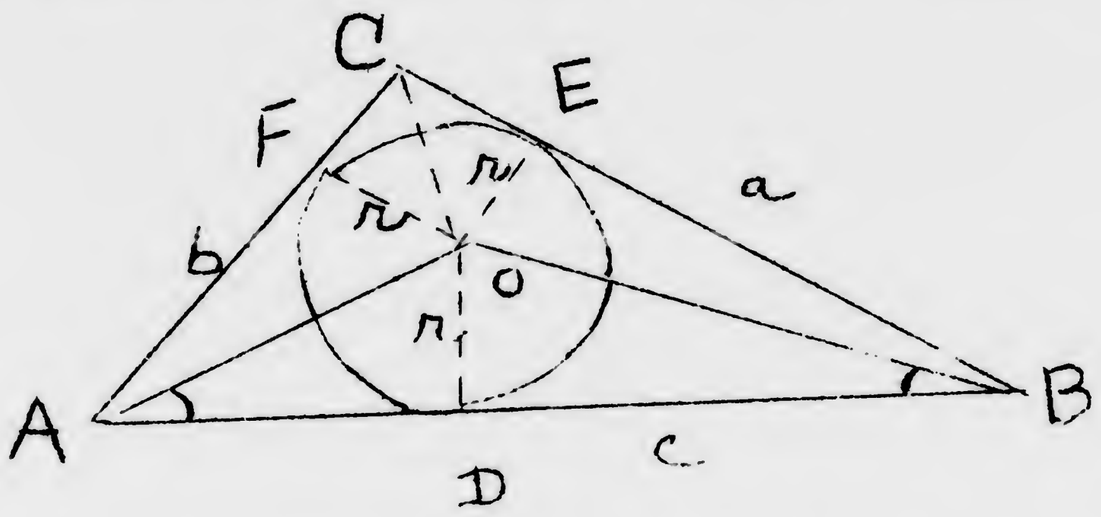
Suggestion $\angle AOD = \angle ACB$

TIGHT BINDING

31. Find the areas of the circumscribed circles in exercises 10, 11, 12, 13, 14.

We have seen that the law of cosines enables us to solve triangles under Cases III and IV, by using the natural functions; but that this law is not adapted to work with logarithms. We shall now derive formulas adapted to logarithmic methods.

The Half-Angles Formulas. Since the center O of the inscribed circle lies on the intersection of the bisectors of the angles of the triangle A B C, we have from the figure



$$\tan \tfrac{1}{2}A = \frac{r}{AD}, \quad \tan \tfrac{1}{2}B = \frac{r}{BE}, \quad \tan \tfrac{1}{2}C = \frac{r}{CE}.$$

We know from geometry that AD = AF, BE = BD, CE = CF.

Hence, AD + BE + CE = s, half the perimeter.

$$\therefore \quad AD = s - (BE + CE) = s - a$$

$$\text{Hence} \quad \tan\tfrac{1}{2}A = \frac{r}{s-a}, \quad \tan\tfrac{1}{2}B = \frac{r}{s-b}, \quad \tan\tfrac{1}{2}C = \frac{r}{s-c}$$

These half-angle formulas enable us to solve a triangle when its sides are given by the method of logarithms.

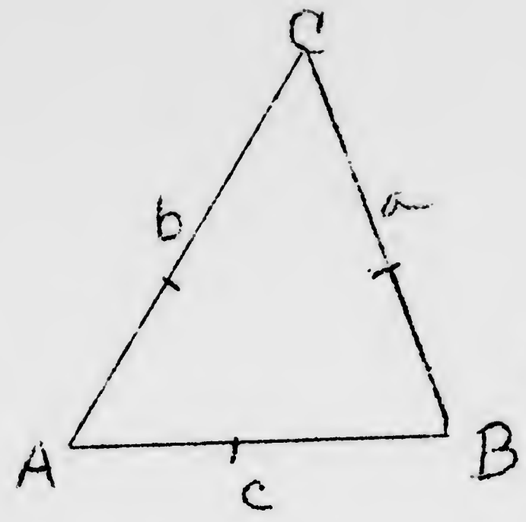
Case IV. The following formulas are used to solve by logarithms a triangle when the three sides are given:

$$\tan\tfrac{1}{2}A = \frac{r}{s-a}, \text{ etc.} \quad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \quad K = rs.$$

Example.

Given,

a	= 2.51
b	= 2.79
c	= 2.33
2s	= 7.63
s	= 3.815
s - a	= 1.305
s - b	= 1.025
s - c	= 1.485
(Check) 2s	= 7.63



Find,

A	= 57° 52′ 48″
B	= 70° 17′ 28″
C	= 51° 50′ 8″
A+B+C	= 180° 00′ 24″
K	= 2.7528

Estimated results: A = 58°, B = 70°, C = 52°

Formulas: $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$,

$\tan\frac{1}{2}A = \frac{r}{s-a}$, etc. $K = rs$

$$\log r = \tfrac{1}{2}\left[\log(s-a)+\log(s-b)+\log(s-c) - \log s\right]$$

log(s-a)	= .11561
log(s-b)	= .01072
log(s-c)	= .17173
	20.29806 - 20
log s	= .58149
log r	= ½(19.71657 - 20
log r	= 9.85829 - 10
log (s-a)	= .11561
log tan ½A	= 9.74268 - 10
½A	= 28° 56′ 24″

log r	= 9.85829 - 10
log (s-b)	= .01072
log tan ½B	= 9.84757 - 10
½B	= 38° 8′ 44″

log r	= 9.85829 - 10
log (s-c)	= .17173
log tan ½C	= 9.68656 - 10
½C	= 25° 55′ 4″

log r	= 9.85829 - 10
log s	= .58149
log K	= 10.43978 - 10
K	= 2.7528

Exercise

Solve the following triangles and find their areas:
(Make a complete form before using the tables)
Record the time spent upon each exercise.

1.	a = 86.5 b = 73.2 c = 90.1	2.	a = .342 b = .581 c = .666	3.	a = 1.302 b = 0.870 c = 0.903
4.	a = 31.24 b = 25.07 c = 30.68	5.	a = 1760 b = 2020 c = 2805	6.	a = 10.70 b = 8.65 c = 8.20
7.	a = 0.062 b = 0.085 c = 0.106	8.	a = 92.5 b = 86.3 c = 165.7	9.	a = 135.24 b = 210.73 c = 206.05

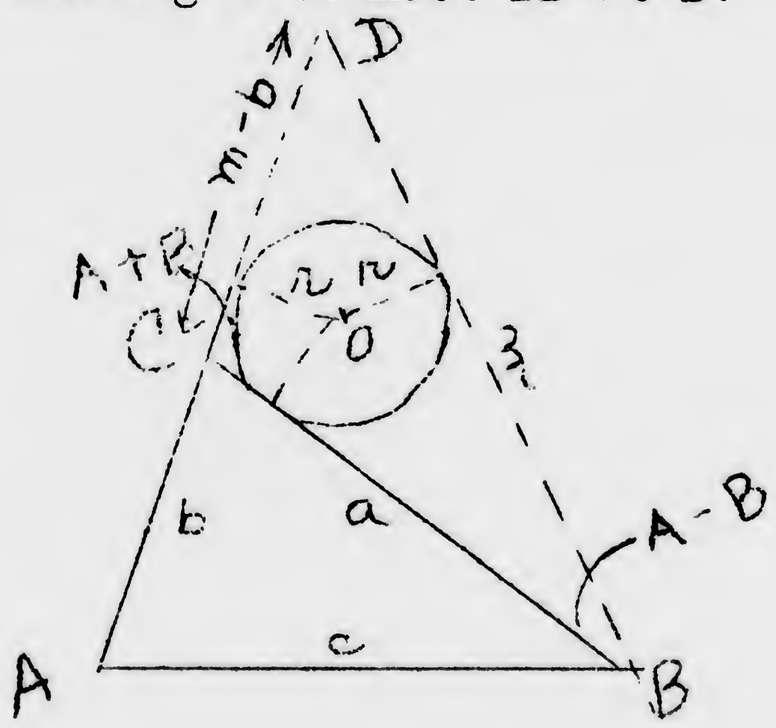
10. By combining the formulas for r and $\tan\frac{1}{2}A$, show that

$$\tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

11. Use the formula in exercise 10 to find one angle in exercises 7, 8, and 9. This formula is convenient when one angle is required.

12. Two sides of a parallelogram are 52.3 and 41.6, and one of the diagonals is 78.4. Find the other diagonal and the angles of the parallelogram.

The Law of Tangents. Let ABC be any triangle. On any side c construct the isosceles triangle ABD by making the angle ABD equal to angle A, and producing b to meet BD at D.



TIGHT BINDING

In the triangle BDC one angle is A + B and another is A - B, the opposite sides are m and m-b.

By the formulas of the half angles we have:

$$\tan\tfrac{1}{2}(A - B) = \frac{r}{s-(m-b)}, \quad \text{and}$$

$$\tan\tfrac{1}{2}(A + B) = \frac{r}{s - m}$$

By division, $$\frac{\tan\frac{1}{2}(A - B)}{\tan\frac{1}{2}(A + B)} = \frac{s - m}{s - (m - b)}$$

But $s = \frac{1}{2}a + \frac{1}{2}m + \frac{1}{2}(m - b) = m + \frac{1}{2}(a - b)$.

Hence, $s - m = \frac{1}{2}(a - b)$, and $s - (m - b) = \frac{1}{2}(a + b)$.

Substituting we have,

$$\frac{\tan\frac{1}{2}(A - B)}{\tan\frac{1}{2}(A + B)} = \frac{a - b}{a + b}, \text{ or } \frac{\tan\frac{1}{2}(B-A)}{\tan\frac{1}{2}(B+A)} = \frac{b - a}{b + a}$$

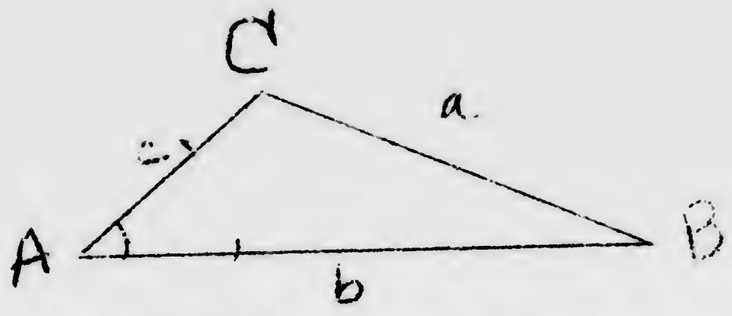
(Note. In the figure A is acute. The same relation can be proved when A is obtuse.)

That is, The tangent of half the difference of any two angles of a triangle is to the tangent of half their sum as the difference of the opposite sides is to the sum of those sides.

Case III. This is the Law of Tangents which may be used instead of the Law of Cosines to solve triangles of Case III, when two sides and the included angle are given. It is adapted to logarithms.

Example.

Given,
b = 0.312
c = .917
A = 33° 7′ 9″
c-b = .605
c+b = 1.229



Scale 1 cm = .3

Find
B = 14° 34′ 24″
C = 132° 18′ 28″
a = .6775

C+B = 180 - A = 146° 52′ 51″

$\frac{1}{2}(C+B)$ = 73° 26′ 26″

Formula: $\frac{\tan\frac{1}{2}(C-B)}{\tan\frac{1}{2}(C+B)} = \frac{c-b}{c+b}$.

$\log\tan\frac{1}{2}(C-B) = \log\tan\frac{1}{2}(C+B) + \log(c-b) - \log(c+b)$

log $\tan\frac{1}{2}(C+B)$ = .52674
log (c - b) = 9.78176 - 10
10.30850 - 10
log (c+b) = .08955
log $\tan\frac{1}{2}(C-B)$ = .21895
$\frac{1}{2}(C-B)$ = 58° 52′ 2″
$\frac{1}{2}(C+B)$ = 73° 26′ 26″
C = 132° 18′ 28″
B = 14° 34′ 24″

To find a,

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$a = \frac{b \cdot \sin A}{\sin B}$$

log b = 9.49415 - 10
log sin A = 9.73750 - 10
19.23165 - 20
log sin B = 9.40074 - 10
log a = 9.83091 - 10
a = .6775

Check by Mollweide's formula. (Remember that A + B + C = 180° is not a check to solutions by the Law of Tangents)

$$a-b = \frac{c \sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C},$$

a - b = .3655
(A - B) = 18° 32′ 45″
$\frac{1}{2}$(A - B) = 9° 16′ 23″
$\frac{1}{2}$C = 66° 9′ 14″

log c = 9.96237 - 10
log sin $\frac{1}{2}$(A-B) = 9.20721 - 10
19.16958 - 20
log cos $\frac{1}{2}$C = 9.60668 - 10
log (a-b) = 9.56290 - 10
a - b = .3655

Checks.

Exercise.

Solve the following triangles and find the areas. Time yourself. The average student should solve each problem in less than 25 minutes:

1. a = 67.3, b = 50.8, C = $44^\circ\ 22'$.

2. a = 123.5, c = 110.2, B = $78^\circ\ 8'$.

3. b = .703, c = .562, A = $104^\circ\ 20'$.

4. a = 1.238, b = 2.006, C = $65^\circ\ 50'\ 20''$.

5. b = .807, c = 1.436, A = $125^\circ\ 25'\ 25''$.

6. a = 4105, c = 5280, B = $25^\circ\ 25'$.

7. b = .079, c = .138, A = $37^\circ\ 33'\ 42''$.

8. a = 305.4, b = 250.7, C = $53^\circ\ 45'$.

9. b = 42.37, c = 36.00, A = $81^\circ\ 12.4'$.

General Exercise.

Solve the following triangles and find their areas. Do not use logarithms.

1. a = 15, c = 20, B = $50^\circ 20'$.

2. a = 12, b = 16, c = 25.

3. b = 40, c = 55, A = $110^\circ\ 25'$.

4. a = 150, b = 200, c = 130.

5. a = 75, b = 110, C = $18^\circ\ 10'$

6. a = 2.4, b = 1.6, c = 3.2

In solving the following triangles use logarithms:

7. a = .015
   b = .024
   B = $134^{o}\ 11'\ 4''$

8. a = 567.89
   c = 663.99
   C = $51^{o}\ 20'$.

9. b = 7252
   B = $45^{o}\ 22'\ 20''$
   C = $33^{o}\ 54'\ 14''$

10. a = 8.706
    b = 7.902
    c = .916.

11. b = 609
    c = 641
    A = $18^{o}10'\ 50''$.

12. c = 91.06
    b = 77.04
    B = $41^{o}\ 13'\ 0''$.

13. a = 96.236
    B = $71^{o}\ 46.9'$
    C = $44^{o}\ 58.6'$.

14. a = 83.856
    b = 85.153
    c = 65.696 .

15. b = .08
    c = .153
    A = $41^{o}\ 24.6'$.

16. b = 1.4532
    c = .45296
    A = $74^{o}19'\ 38''$

17. b = 76.438
    B = $143^{o}\ 18'\ 31''$
    C = $13^{o}\ 34'\ 26''$.

18. a = 7.6328
    b = 24.359
    c = 19.555.

19. b = 253.6
    c = 928.5
    B = $50^{o}\ 25'$.

20. a = .15876
    b = .0998
    c = .0654.

21. a = 1.123
    c = - 2.034
    B = $72^{o}\ 15'\ 19''$.

The Applications of Trigonometry.

Trigonometry was practical in its origin. It was invented because it was necessary for astronomy. It is likewise necessary for land-surveying, navigation, mechanics, and engineering. In this section we shall consider only a few of its innumerable practical applications.

I. Land Surveying.

One of the most practical applications of triginometry is in land surveying.

To survey land is to measure and compute distances and angles in order to locate boundaries, find areas and make maps.

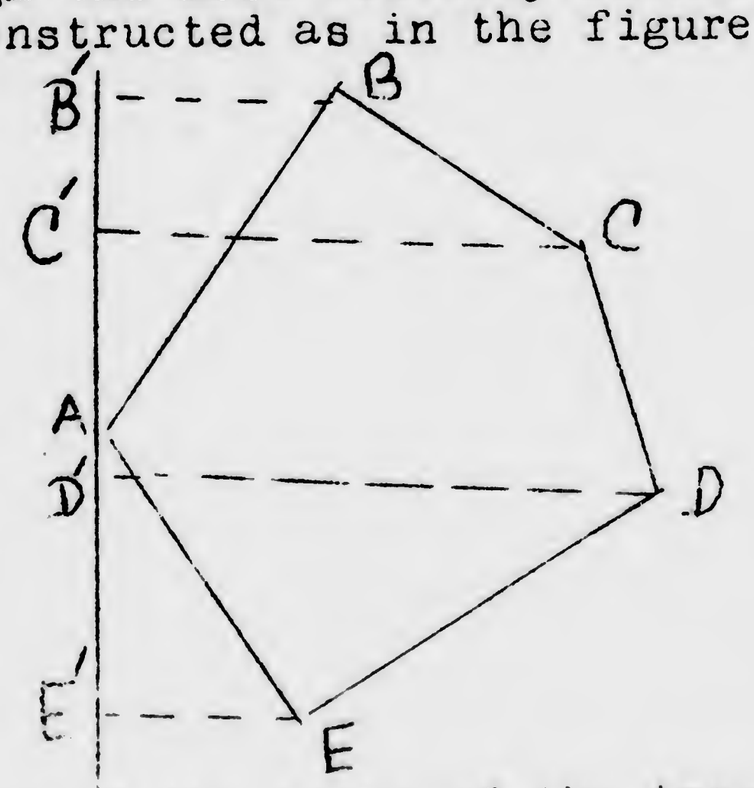
The surveyor's instruments are a transit, a tape or chain, chain pins, rods, and poles.

In a survey distances are measured along the horizontal, and areas are computed as if the land were flat, or projected on a horizontal plane.

In measuring city lots surveyors use feet and square feet. In measuring larger pieces of land they use feet, rods, and acres.

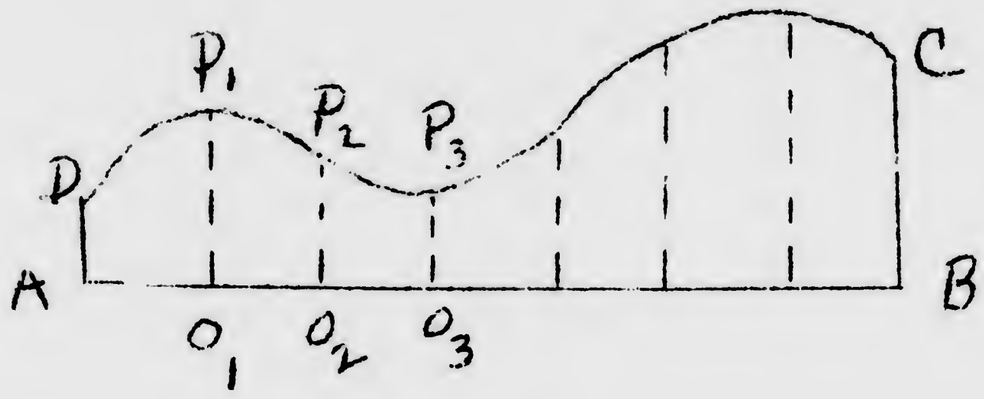
Areas.

To find the area of a piece of land bounded by straight lines, the figure is divided into triangles, rectangles, and trapezoids. For example, to find the area of a piece of land represented by the polygon A B C D E, a north and south line is passed through the most westerly corner, and trapezoids and triangles are constructed as in the figure.



Find the sum of the areas of the trapezoids E′E D D′, D′D C C′, and C′C B B′. From this subtract the sum of the areas of the triangles E′E A and A B B′. The result is the area of the field.

When one side of a field is bounded by a curved line due to a river or a curved road, a line is usually run cutting off the irregular part of the field, as A B in the figure.



The offsets A D, O_1P_1, C_2P_2, O_3P_3 ----- B C, are measured at equal intervals on the line A B. The approximate area is found if each of the strips is considered a trapezoid. The method used is as follows:

1. Add all the offsets.
2. Subtract from the result half the sum of the extreme offsets.
3. Multiply the remainder by the common interval.

For details concerning surveying, the student is referred to books on practical surveying.

Exercise.

1. Make a brief report on the government survey of public lands.

2. Give a brief description of the transit and its use.

3. How many acres are there in a trianglar field whose sides are 12 chains, 15 chains, and 20 chains? ($16\frac{1}{2}$ ft. = 1 rod; 4 rods = 1 chain; 160 sq. rd. = 1 acre.)

4. The area of a rectangular field is 160 acres. One side is 22.30 chains; find the other side.

5. The sides of a parallelogram are 52 feet and 35 feet. They include an angle of 120°. Find the area.

6. In the trapezoid A B C D, angle A = 100°, angle B = $41^\circ 20'$, the parallel sides A B and C D are 56. 2 feet and 34. 6 feet respectively. Find the area.

7. The diagonals of a four sided field are 25 rods and 30.5 rods. They intersect at an angle of $65^\circ 35'$. Find the area of the field.

8. A farmer wishes to buy a triangular field on the corner of two roads which meet at an angle of 68°. The frontage on one road is 44.2 rods and on the other 52.6 rods. What will it cost him at $125.50 per acre?

9. A surveyor measured a triangular piece of land. His notes gave A B = 285.1 ft., B C = 234.7 ft., C A = 165.4 ft., angle A = $55^{\circ}\ 3'$, angle B = $40^{\circ}\ 21'$, angle C = $84^{\circ}36'$. Are his measurements correct?

10. Draw to scale and find the area of the field whose boundaries are:

Course	A B	B C	C D	D A
Bearing	N 45°E	S 75°E	S 15°W	N 45°W
Distance	100 ft.	115.5 ft.	182.1 ft.	191.1 ft.

11. Draw to scale and find the area of the field whose boundaries are:

Course	A B	B C	C D	D E	E A
Bearing	N 41°E	S 70°E	S 29°E	West	N $94\frac{1}{4}^{\circ}$ W
Distance	300 ft.	400 ft.	720 ft.	750 ft.	480 ft.

12. A surveyor finds that one tree is 225 ft. N $16^{\circ}\ 20'$ E from him, and another tree is 315 feet from him N $51^{\circ}\ 40'$ W. How far apart are the trees?

13. A man wished to buy a triangular lot on the corner of two streets which do not meet at right angles. The frontage on one street is 400 feet and on the other 275.5. The back line of the lot is 500 feet long. What will it cost him at $1.25 per square foot? Find the angle between the streets.

14. Find the radius of the largest cylindrical gas tank that can be constructed on a triangular lot whose sides are 30, 40, and 55 feet respectively. Locate the center of its circular base.

15. To find the distance between two objects separated by an obstruction, a surveyor chose a point C and measured C A = 1516 feet and C B = 1258 feet, and the included angle = $100^{\circ}\ 20'$. What is the distance between the objects?

II. Measurement of Heights and Distances.

One of the most important applications of trigonometry is the measurement of the heights and distances of objects which cannot be actually measured.

The following list will give the student an idea of the variety of problems which depend for their solution upon triginometry. The student should illustrate every problem by a figure, analyze it carefully, and explain the method of its solution.

1. To find the distance of an inaccessible point A from a point B, a base line B C = 350 feet and the angles A B C = $54^\circ\ 25'$ and ACB = $42^\circ\ 33'$ are measured. Find A B and the shortest distance from A to B C.

2. To find the distance between two points A and B separated by a swamp, a surveyor chose a station C and measured C A = 325.4 ft., C B = 415.7 ft., and the angle A C B = $77^\circ\ 25'$. Find the distance from A to B.

3. To find the height of a monument a man observed from a certain point that the angle of elevation of the top of the monument was $65^\circ\ 15'$. He retired 100 feet and observed that the angle of elevation was $50^\circ\ 45'$. What is the height of the monument?

4. A person standing on the bank of a river observes the angle subtended by a tree on the opposite bank to be $53^\circ\ 30'$, and when he retires 50 feet from the river's bank he observes the elevation to be $40^\circ\ 10'$. Find the width of the river.

5. To find the slope of a railroad embankment, one end of a pole 12 feet long was placed on the level ground 6 feet from the foot of the embankment, and the other end was found to fall at a point 8.6 feet up its face. What angle does the embankment make with the horizontal plane

6. A person wishes to find the width of a river. Illustrate the problem by a figure, and describe the necessary measurements and the method of solution.

7. Illustrate by a figure the problem of finding the height of the tower of Old Main, and describe the necessary measurements and the method of computation.

8. A person wishes to find the distance of a boat from a lighthouse on the shore. Illustrate the problem by a figure and describe the necessary measurements and computations.

9. Show how a surveyor can find the height of a tree on an inclined plane.

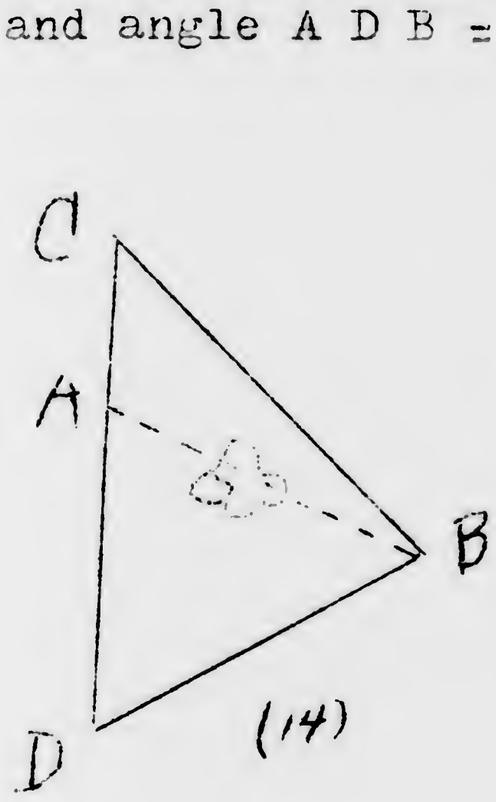
10. Show how a person can find from the sea the height of a castle on the top of a clift on the shore.

11. Show how to find the angle between two walls.

12. Show how a surveyor can find the distance between two inaccessible objects by measuring angles at the ends of a known line.

13. The sides of a triangle are in the ratio 4:5:7. Find the angles. Is the triangle determined? Can you find its area?

14. Two points A and B (see figure) are on opposite sides of a stream and invisible from each other. A straight line C D is run through A and the following measurements taken: C A = 820 ft., A D = 492 ft., angle A C B = $48^\circ 20'$, and angle A D B = 59 12 . Find A B.



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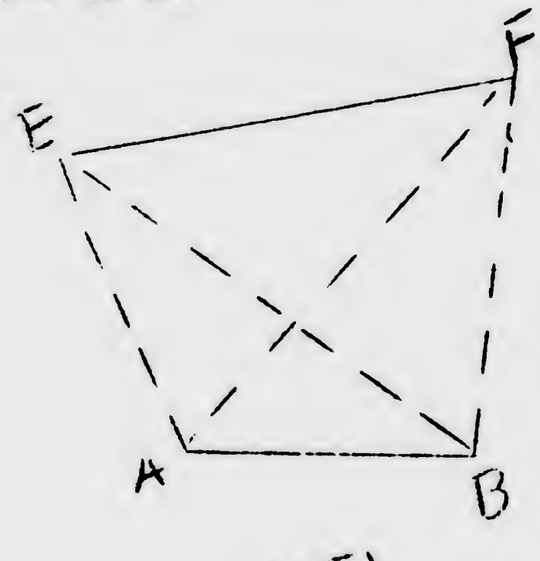


(15)

15. In order to find the distance between two inaccessible objects E and F, a base line A B = 500 yards was laid. Four angles were measured as follows: A B E = $38^\circ 10'$, A B F = $96^\circ 40'$, B A E = $118^\circ 20'$, and B A F = $47^\circ 15'$. Find E F.

16. A surveyor measured a triangular piece of land. His notes gave A B = 475 ft., B C = 165 ft., and angle C A B = $29^\circ 12'$. Show that these notes must be wrong in some way.

17. From the top of a house 50 feet high the angle of elevation of the top of a steeple is $16^\circ 27' 40''$; at the bottom of the house it is $24^\circ 15' 35''$. Find the height and distance of the steeple.

18. A picture 4 feet long is hung vertically on a wall, with its lower end 8 feet from the floor. Under what visual angle is it seen by a person 5 ft. 6 in. tall when standing 7 feet from the wall?

19. The parallel sides of a trapezoid are 23.2 inches and 33.2, and the angles at the ends of the longer side are $70^\circ\ 18'$ and $50^\circ\ 32'$. Find the other sides and the area of the trapezoid.

20. The parallel sides of a trapezoid are 3 feet 8 inches and 4 ft. 6 in., and the angles adjacent to the shorter side are 120° and 100°; Find the other sides and the area.

21. Two sides of a triangle are 5.34 ft. and 4.68 ft. and the difference of the angles opposite these sides is $16^\circ\ 28'\ 15''$. Solve the triangle.

22. Two sides of a parallelogram are 152.3 and 106.8, and one diagonal is 180.6. Find the other diagonal and the area of the parallelogram.

23. Two observers stationed three miles apart on opposite sides of an airplane in the same vertical plane with them find its angles of elevation to be $37^\circ\ 20'$ and $49^\circ\ 42'$. What is the height of the airplane and its distance from each observer?

24. The diagonals of a parallelogram are 9 ft. and 14.5 ft., and one side is 7 ft. Find the angles of the parallelogram and its area.

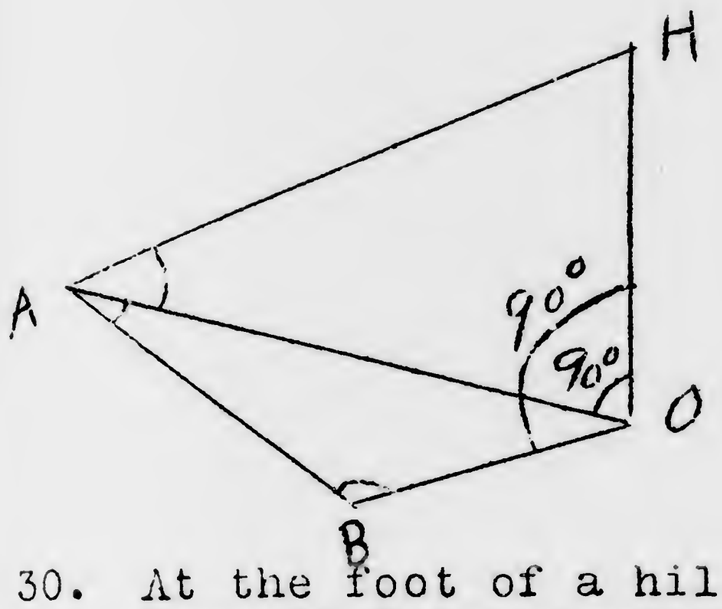
25. From the top of a lighthouse 150.2 ft. above sea level, the angles of depression of two ships in the same line with the lighthouse are $12^\circ\ 5'$ and $10^\circ\ 26'$. How far apart are the ships?

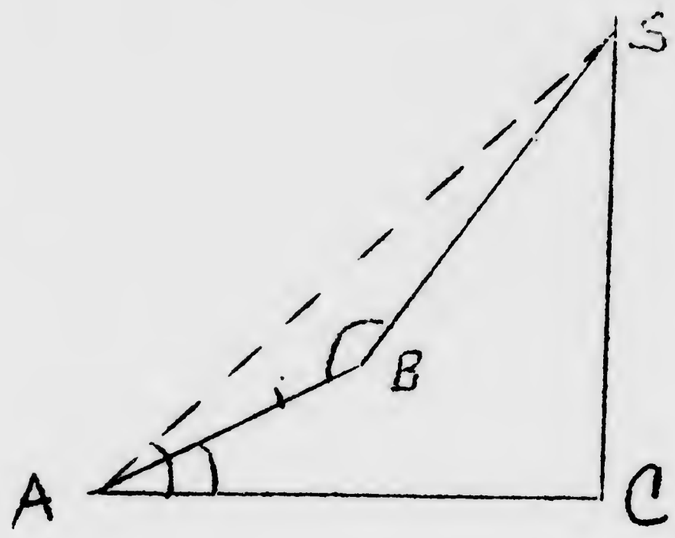
26. From the top of a lighthouse 100 feet high, standing on a cliff, the angle of depression of a ship was $4^\circ\ 27'$, and at the bottom of the lighthouse, the angle of depression was $3^\circ\ 10'$. Find the horizontal distance, of the ship and the height of the cliff.

27. A tower makes an angle of $107^\circ\ 25'$ with the slope of the hill on which it stands. At a distance of 100 feet measured down hill from the base of the tower, the angle of elevation of the top of the tower is $47^\circ\ 10'$. Find the height of the tower.

28. A triangle A B C on a horizontal plane has $A = 55^\circ\ 23'$, $B = 73^\circ\ 5'$ and $c = 250$ ft. Find the height of a tree standing at C if the elevation angle of its top from A is $10^\circ\ 15'\ 30''$.

29. In order to find the height of a mountain O H, a base line A B = 1000 ft. was taken and the following measurements were made: angle B A O = $63^{\circ}\ 20'$, angle O A H = $9^{\circ}\ 50'$, and angle A B O = $75^{\circ}\ 45'$. A, B, and O are in a horizontal plane. Find the height of the mountain.



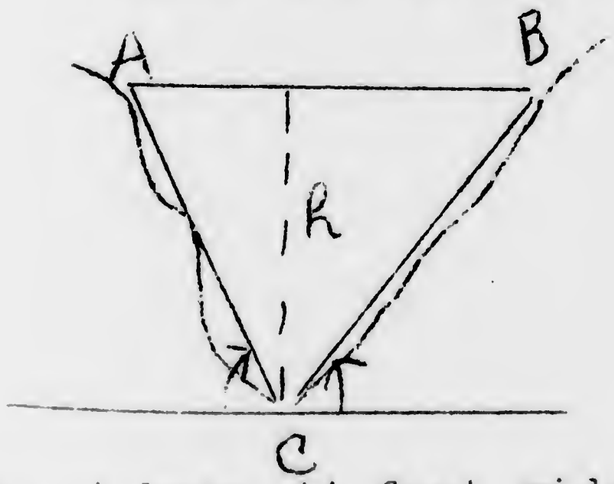


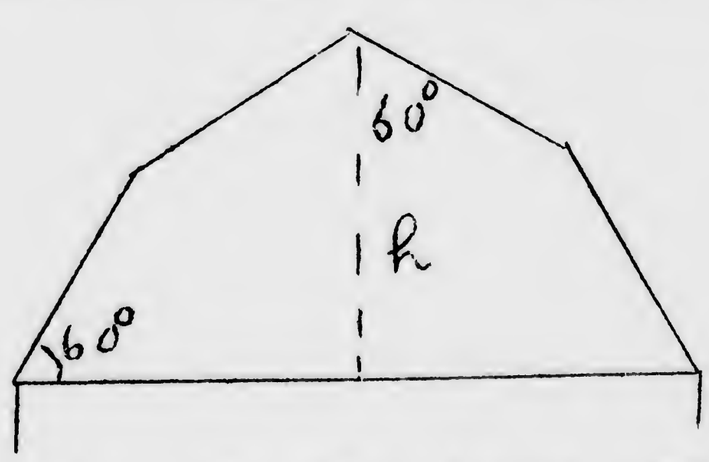
30. At the foot of a hill A the angle of elevation of the summit S is observed to be $42^{\circ}\ 23'$, and the angle of elevation of a point B, 800 feet up the hill, $27^{\circ}\ 20'$. The angle A B S is $143^{\circ}\ 22'$. Find the height of the hill.

31. At the foot of a hill the angle of elevation of its top is 40°. At a point 650 feet up the hill at a slope of $25^{\circ}\ 40'$ inclination, the angle of elevation of the top of the hill is 50°. Find the height of the hill if the two points of observation are in the same vertical plane with the top of the hill.

32. The sides of a triangle are 0.568, 0.732 and 0.608 feet. Find the length of the median upon the longest side, the length of the altitude upon the shortest side, and the length of the bisector of the smallest angle.

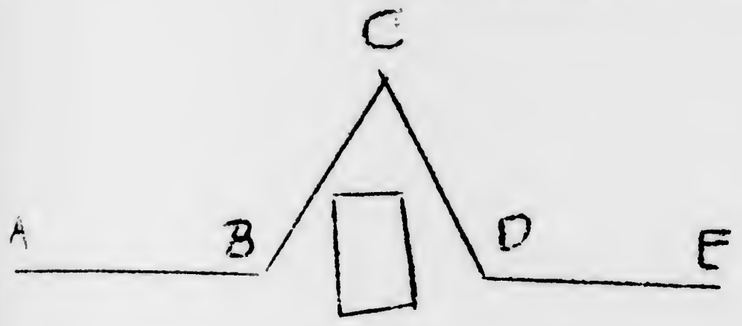
33. An engineer wanted to build a bridge across a valley from A to B, two points on the same horizontal line, and support it by a pier at C. From C the angle of elevation of A = $64^{\circ}\ 20'\ 25''$, and that of B = $51^{\circ}\ 36'\ 40''$. The distance A B is 350.2 feet. Find the height of the pier.

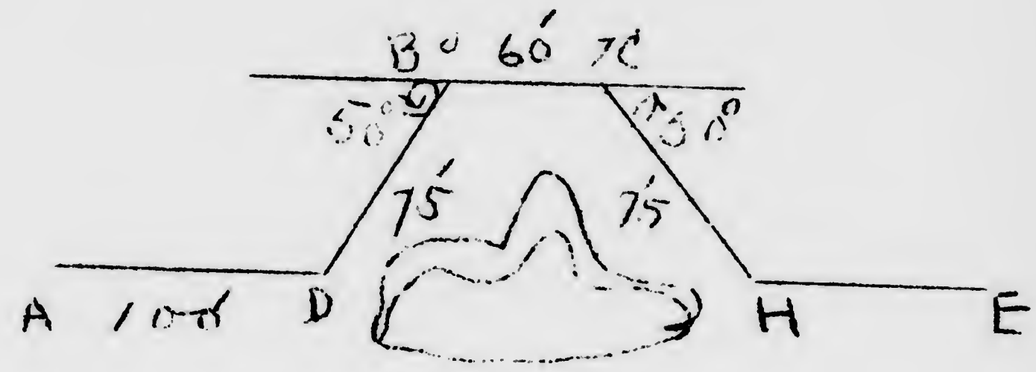




34. A barn 44 feet wide has a gambrel roof. The lower rafter makes an angle of 60° with the horizontal. The upper rafter makes an angle of 60° with the vertical. The lower and upper rafters are equal in length. Find the length of the rafters and the height of the ridge.

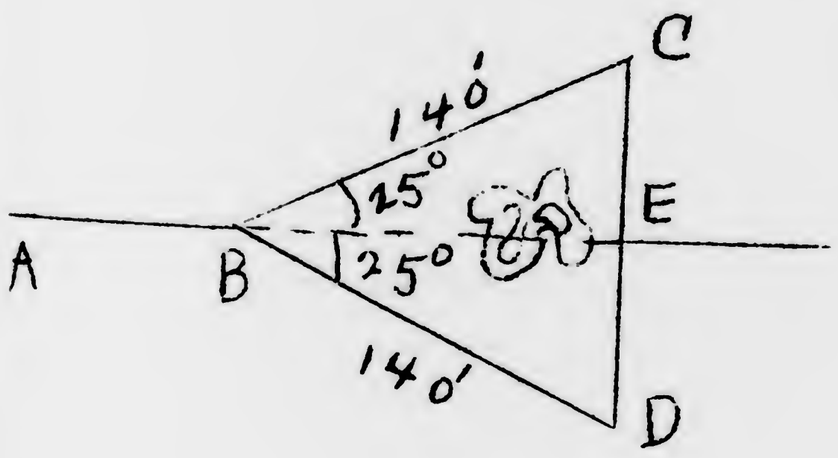
35. It is sometimes desirable in surveying to extend a line when an object interferes as in the adjoining figure. Show that this can be done by means of the broken line A B C D E. What measurements are necessary?





36. In extending the straight line A D in the figure, a surveyor found that a building interfered. He took measurements as shown. Find the distance on a straight line from D to H.

37. In order to extend the line A B, a surveyor found that a pile of stones interfered. He took measurements as shown. Find C E and B E.



38. O and P are two inaccessible points both of which are visible from one point A only. Show what measurements may be made so that the distance O P can be determined.

39. Find the number of square yards of cloth in a conical tent with a circular base if the inclination of the side is 54° and the center pole is 15 ft. high.

40. Find the number of square feet in the conical roof of a silo if the vertical angle is 102° and the apex of the roof is 12½ feet above its base. How many cubic feet of concrete are in the roof if it is six inches thick?

TIGHT BINDING

41. A lighthouse was observed from a ship to bear N 42° 15′ E. After the ship sailed due north 6 miles the lighthouse bore N 65° 25′ E. Find the distance of the lighthouse from the ship at both positions.

42. Two steamers start at the same time from the same point and sail one due east at the rate of 12.3 miles an hour, and the other south east at the rate of 14.1 miles an hour. What is the distance between them at the end of 45 minutes, and what is the bearing of the first from the second?

43. A ship starts from A due west at the rate of 12 miles an hour. A dispatch boat leaves at the same time a point 20 miles southeast of A. If the boat has a rate of 25 miles an hour, what should be the direction of its course that it may meet the ship? How far from A and B would they meet, and at what time?

44. A ship sailing southwest at the rate of 16 miles an hour observes at 8 o'clock a lighthouse bearing S 25°W. At 10:30 it bears N 70° W. Find the distance of the boat from the lighthouse at both times.

Note: The path of light in a homogeneous medium like air is a straight line. But if a ray of light strikes a smooth metallic surface or a mirror, its direction is changed in accordance with the law that the angle of incidence is equal to the angle of reflection. In general, the image of an object in a plane mirror is the same size as the object, and as far behind the mirror as the object is in front.

45. A man 5 ft. 8 in. tall is standing on the bank of a river. He observes that the angle of elevation of the top of a tree on the bank is 36° 20′, and the angle of depression of the image of the top of the tree in the water is 46° 20′. Find the height of the tree and the width of the river.

46. When light passes obliquely from one transparent medium into another, it undergoes a change in direction at the boundary surface of separation between the two media. This change in the course of light is called refraction. The index of refraction M is d termined by the relation

$$M = \frac{\sin i}{\sin i'}$$

Where i and i′ are the angles between the ray of light in the rarer and denser mediums respectively, and the line vertical to the separating surface. Experiments show that this ratio is always the same for the same two substances, no matter what the angle of incidence may be. The ratio is different for different substances or different kinds of the same substance.

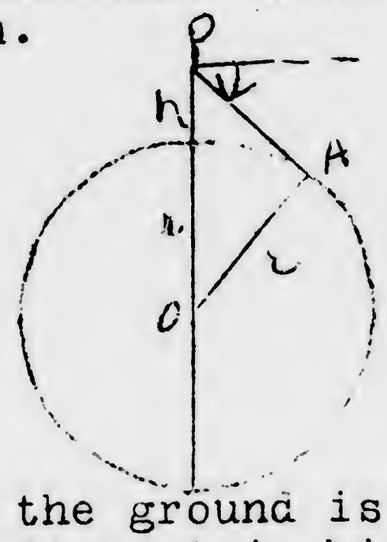
Find i′ when M = 1.125 and i = $22^{o}\,30'$.

Find i and i′ when M = 1.3 and the deflection is $15.^{o}$

47. The shadow of a tree lengthened 20 feet from the time the sun was 62^{o} above the horizon to the time when it was $40^{o}\,30'$ above the horizon. Find the height of the tree.

48. From a mountain 4000 feet high, overlooking the sea, an observer looked toward the horizon. Find the distance and dip of the horizon.

(P A is ⊥ r; r = 3960 miles)



49. A man whose eye is 5 ft. 8 in. from the ground is standing on the sea shore. How far distant is his horizon?

50. Hor high must an aviator rise to see the horizon 40 miles distant?

51. Find the height of a lighthouse above the sea level when it begins to be seen at a distance of 14 miles?

52. Two lighthouses, one 100 feet high and the other 80 feet high, are just visible from each other over the water. How far are they apart?

53. From an airplane at an elevation of 4 miles the dip of the sea-horizon is observed to be $2^\circ 33' 50''$; find the diameter of the earth and the distance of the horizon from the airplane.

54. To determine the dip of a geological stratum that is under ground, three holes are bored at three angular points of a horizontal square of side 40 feet. The depths at which the stratum is struck are 20 feet, 25.6 feet, and 32 feet. Find the dip of the stratum. Show that

$$\tan D = \frac{\sqrt{(p - q)^2 + (q - r)^2}}{a},$$

Where D = dip, p, q, and r are the depths, and a is the side of the square.

55. In a monument, the outer diameter of a winding stairway is 12 feet. Find the angle of the spiral formed by the outer end of the steps if the stairway makes one turn in ascending 18 feet. Find the angle of the spiral formed by the inner ends of the steps if the steps are 4 feet long.

$$\tan A = \frac{\text{circumference of cylinder}}{\text{lead, (advance per turn)}}.$$

Ans. $64^\circ 29'$; $34^\circ 55'$.

56. At a certain time of the day a stick 5 feet high casts a shadow 3 feet long. At the same time a pole making an angle of 76° with the horizontal casts a shadow 32 feet 5 inches long. Find the length of the pole.

57. When the sun is vertically above the equator the shadow of a pole 20 feet high is 12 feet long. What is the latitude of the place where the pole stands?

Note. The latitude of a place is the complement of the angle of elevation of the sun at noon when the sun is above the equator.

58. At a certain place the shortest shadow of a pole 50 feet high at noon is 40 feet, and its longest shadow at noon is 90 feet. What is the latitude of the place?

59. One method of determining the height of the earth's atmosphere is through the observation of meteors. A shooting star is observed (as soon as it begins to glow) from two stations ten miles apart and in the same vertical plane with it. The angle of elevation of the star at one station is $85^\circ 24'$ and at the other $83^\circ 47'$. Find the height of the star at the instant of observation.

60. The earth subtends an angle of $17\frac{3}{4}''$ at the sun. Find the distance of the sun from the earth, the radius of the earth being 3960 miles.

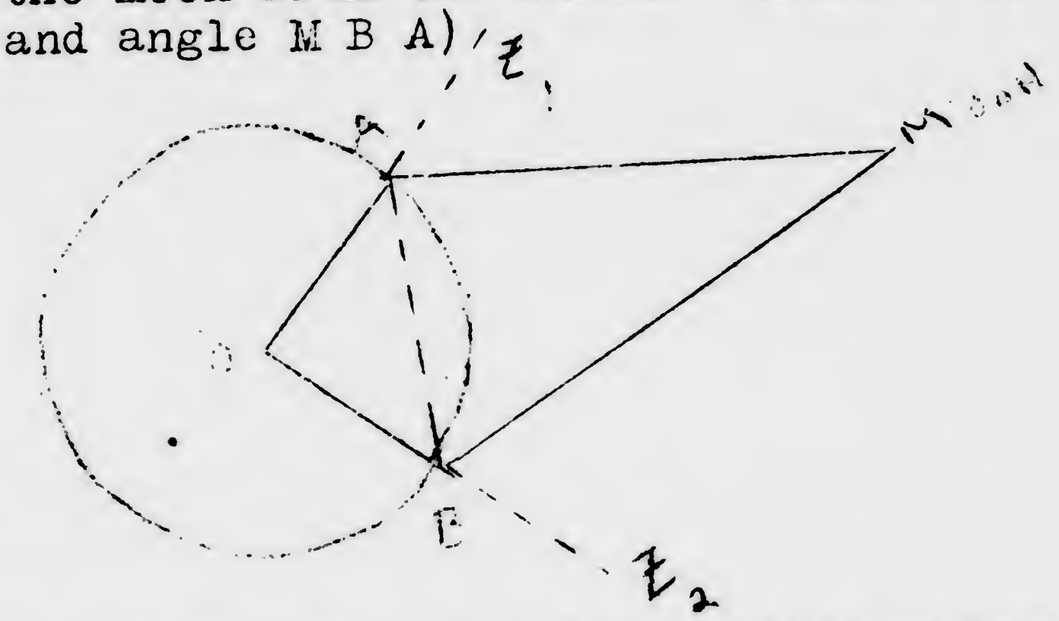
61. At a certain time the distances of the earth and Mars from the sun were respectively 91.1 and 138.6, the angle E S M between their directions being $152^\circ\ 4'$. Find E M, their distance apart at that time.

62. Two stars A and B are at distances 4.5×10^{12} and 3.8×10^{12} miles from the earth; and the angle between their directions is $50^\circ 20'$. Find their distance apart.

Distance to a Star.

The distance to a planet is found by taking a base line and measuring from each end the angle made between the base line and the line of sight to the planet. The distance to a fixed star is found by using the diameter of the earth's orbit as a base line.

63. Two astronomers on the same meridian one at A in London (Greenwich), latitude N $51^\circ\ 17'\ 10''$, the other at B, Cape Town, latitude S $33^\circ\ 45'\ 46''$, find when the moon is in their vertical plane angle $Z_1 A\ M = 35^\circ\ 2'\ 20''$ and angle $Z_2 B\ M = 51^\circ\ 17'\ 10''$. Find the distance of the moon from the earth. (Find A B, angle M A B and angle M B A)

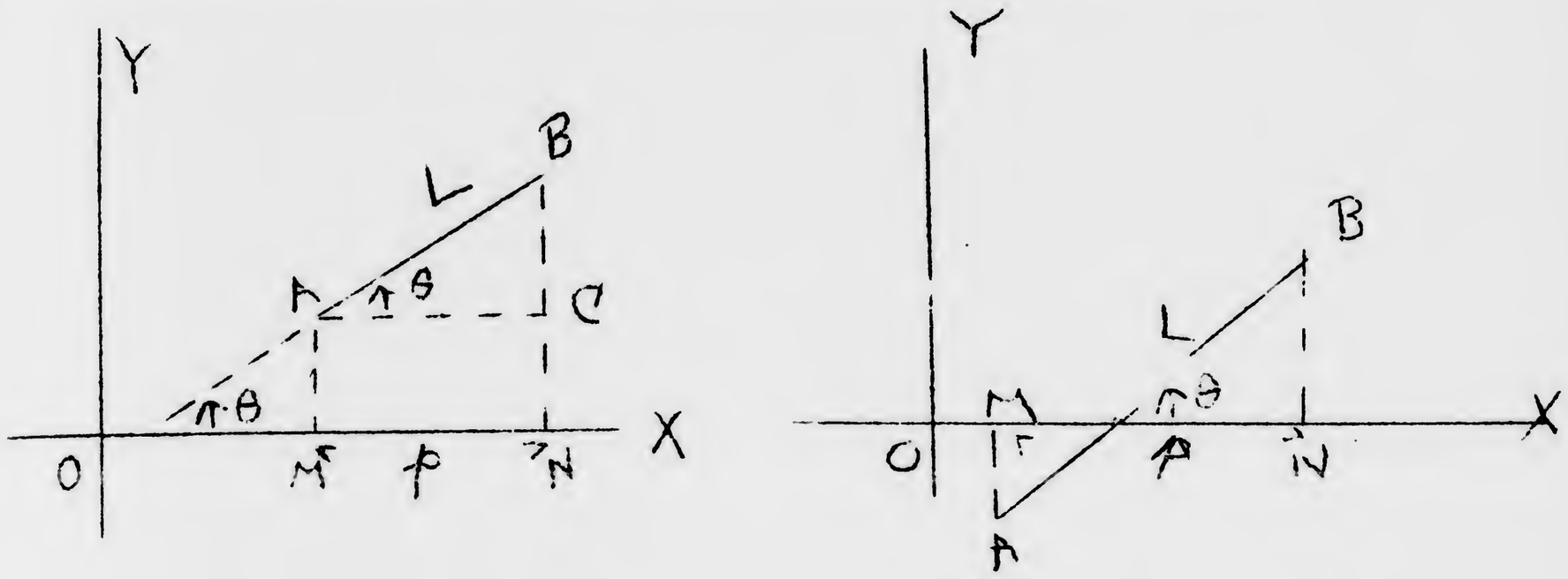


64. Read references on the subject of triangulation and make a brief report.

III. Composition and Resolution of Forces.

Projections.

It is often necessary to consider the projection of a given line segment L upon some other line as O X.



Let A M and B N be perpendiculars from A to B to O X. Then M is the projection of A, N is the projection of B, and M N is the projection of A B on O X.

The projection of a line segment L on a given line is the length between the feet of the perpendiculars dropped from the ends of L upon the given line.

The length of such a projection is easily obtained. For

$$\frac{p}{L} = \cos\theta \qquad \text{or } p = L\cos\theta\ .$$

That is, the projection equals the segment itself times the cosine of the angle between the two lines.

Similarly, the projection of an area upon a plane equals the original area times the cosine of the angle between the two planes.

The projections usually made are upon a horizontal line O X and a vertical line O Y. Hence,

Horizontal or x projection is $x = L\cos\theta$.
Vertical or y projection is $y = L\sin\theta$.

The student should prove that either leg of a right triangle is equal to the hypotenuse times the cosine of the adjacent angle or the hypotenuse times the sine of the opposite angle.

Exercise.

1. Which is longer, a line or its projection?

2. How long is the sum of the projections of the sides of a triangle upon a given line?

3. Find the projection of a line 16.3 inches long upon a line making an angle of $51^\circ\ 20'$ with it.

4. A line 8.62 feet long makes an angle of $42^\circ\ 40'$ with the horizontal. Find its projections on O X and O Y.

5. An inclined plane 12 feet long makes an angle of 8° with the horizontal. Find its height.

6. A farm on a hill side which has a slope of $18^\circ\ 30'$ contains 10.5 acres. What is its projection on the horizontal plane?

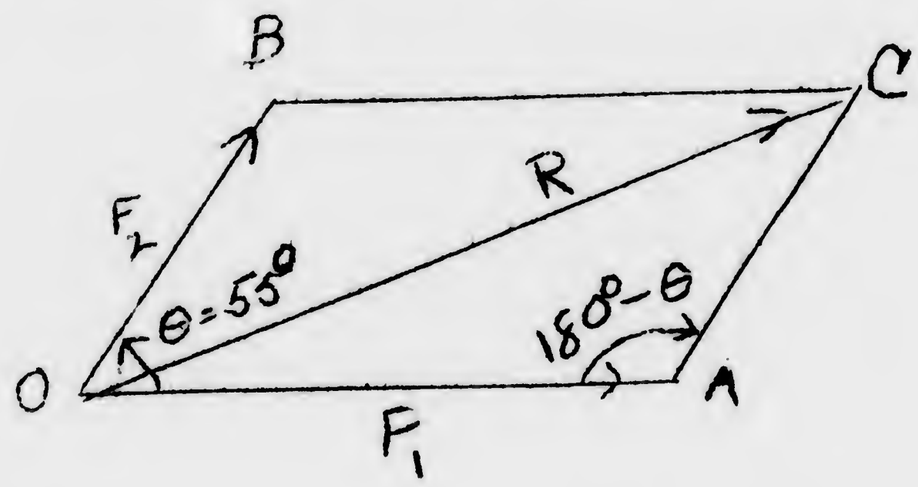
7. A roof $40' \times 20'$ is inclined at an angle of 35°. What area does it cover?

8. The side of an equilateral triangle is c feet long. Find the projection of the altitude upon it.

Representation of Forces and Velocities by Lines.

In physics and engineering line segments are used to represent forces, velocities, and accelerations. The direction of a force or velocity is indicated by an arrow, and its magnitude is indicated by the length of a line drawn to some convenient scale. Lines which represent quantities that have direction as well as magnitude are called Vectors.

Thus the figure represents two forces, one O A of 40 pounds and one O B of 25 pounds. They are acting at the same point O at an angle of 55° to each other.



Resultant.

If these two forces act upon the body at O, the result will be the same as if a single force were applied acting along the diagonal of the parallelogram formed by the two given forces. This construction is called the Parallelogram of Forces. The two forces F_1 and F_2 are called the Components of R, and R is called the Resultant of F_1 and F_2. The magnitude of the resultant is represented by the length of the diagonal. Then, since A C is equal and parallel to O B, it is plain that the two forces and their resultant form a closed triangle. Hence, the relations between two forces and their resultant may be found by solving a triangle, called the triangle of forces.

We have from the law of cosines

$$R^2 = \overline{DA}^2 + \overline{AC}^2 - 2 \cdot OA \cdot A C \cdot \cos A.$$

But $A C = F_2$, and since angle A is the supplement of θ, $\cos \theta = - \cos A$. Therefore we have the

Formula $$R^2 = F_1^2 + F_2^2 + 2F_1 \cdot F_2 \cdot \cos \theta.$$

$$R = \sqrt{(40)^2 + (25)^2 + 2 \times 40 \times 25 \cdot \cos 55^\circ}$$

$$R = \sqrt{3372.2} = 58.1$$

When $\theta = 90^\circ$, the formula reduces to $R^2 = F_1^2 + F_2^2$ (Why?)

The formula may be stated in words as follows:

If two forces act upon the same point, the square of their resultant is equal to the square of the first force plus the square of the second force plus two times the product of the forces and the cosine of their included angle. This principle is true for all values of θ including 0°, 90° and 180°.

If three or more forces act upon a point, their resultant may be found by repeated applications of the parallelogram of forces.

The process of finding the resultant of two or more component forces is called the Composition of Forces.

Exercise.

1. Two forces of 370 lb. and 250 lb. are acting upon a body, making an angle of 30° with each other. Find the magnitude of the resultant and its direction. Find the resultant when the angle is 15°.

2. Two boys are lifting a box by means of ropes that are in the same vertical plane. One boy pulls 62 lb. in a direction 26° from the vertical and the other 86 lb. in a direction 35° from the vertical. Find the weight of the box.

3. Two forces of 200 lb. and 150 lb. act upon a stone, the first at an angle of 60° with the horizontal and the second at an angle of 45° . Find the total force tending to lift the stone vertically, to move it horizontally.

4. A river flows at the rate of 3.6 miles per hour. A man is rowing a boat across the river at the rate of 4 miles an hour. Find the speed of the boat. Where will it land on the opposite bank if the river is one mile wide?

5. Two forces, 18 lb. and 25 lb. act at the same point. Find their resultant (a) if they act in the same direction, (b) if they act in opposite directions, (c) if they act at right angles, (d) if they act at an angle of 120°.

6. A car is running at 12 miles an hour, and a man jumps from it with a velocity of 7 feet per second in a direction making an angle of 45° with the direction of the car's motion. What is his velocity relative to the ground?

7. A ship whose head points northwest is steaming at the rate of 15 miles in a current which flows southwest at the rate of 6 miles? How far will the ship have gone in $3\frac{1}{2}$ hours?

8. A river flows 5 miles an hour, and a man can row his boat (in still water) with a velocity of 3.5 miles. In what direction must he row, if he is to land at a point exactly opposite his atrating point? In what direction must he row so as to cross in the shortest time?

9. Show that if the angle between two given forces is increased, their resultant is diminished.

10. Three forces of magnitude 8, 12, 16 act in the same plane at a given point. Under what conditions will their resultant be equal to zero?

Resolution of Forces.

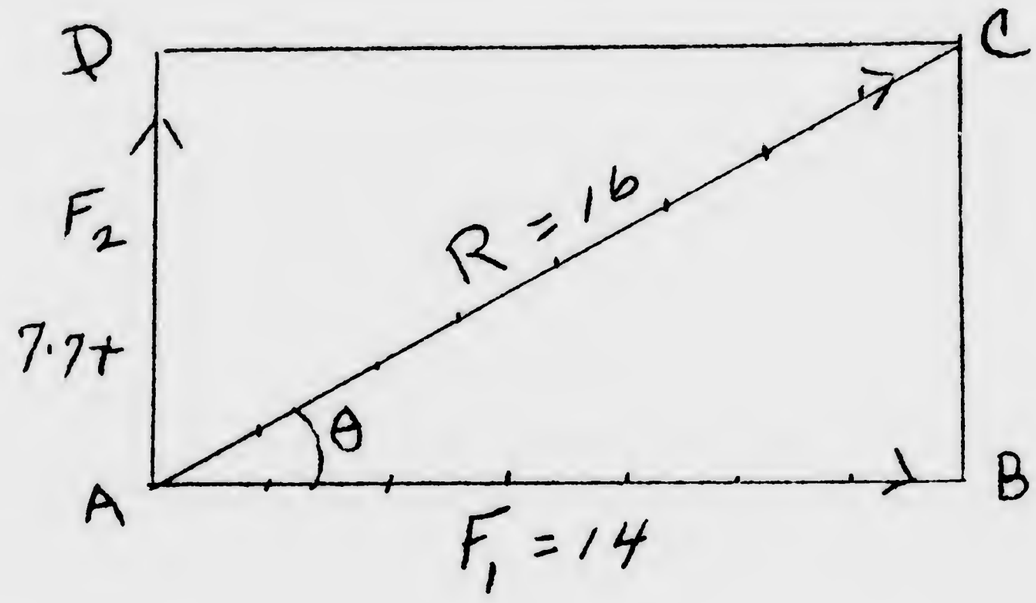
The principle of the composition of forces can be worked backward. If one force is given we can resolve it into components acting in given directions by constructing a parallelogram whose diagonal represents the given force, and whose sides have the given directions of the components.

Every force can be resolved into components in any given directions. The most common case is the resolution of a vector into two components at right angles to each other. Usually one component is horizontal and the other vertical.

Example. Resolve a velocity of 16 miles an hour into two rectangular components.

First. Suppose one of the components required is to be 14 miles an hour, find the other.

Construct the rectangle A B C D with a diagonal of 16 units and one side equal to 14 units.



Then A B and A D are the components of A C, and A D is the component required. Its value obviously is

$$\overline{AD}^2 = \sqrt{16^2 - 14^2} = \sqrt{60} = 7.7+$$

Second. Suppose it is required to find two components one of them making the angle $\theta = 40^\circ$ with A C.

Construct the rectangle. The value of each component is found by multiplying by the cosine of the angle between the given vector and the component required.

From the figure

$$F_1 = R \cos \theta , \quad F_2 = R \cos(90 - \theta) = R \sin \theta .$$

Exercises.

Note. In the solution of these problems it is helpful to make a drawing to scale.

1. A force of 400 lb. is applied to a weight at an angle of 60° to the horizon. What force tends to draw the weight horizontally? What force tends to lift it upward?

2. A force of ten pounds is to be resolved into two forces, one of which is 15 pound, and making an angle of 80° with the given force. Find the other force.

3. A ship is moving in a direction 25° west of south at the rate of 21.4 miles per hour. How far is it moving west? How far south?

4. If the resultant of two forces acting at right angles is 50 pound, and one of them is 20 lb., what is the other? What angles do they make with the resultant?

5. Two horses are drawing a plow with the traces making an angle of 15° with the horizontal. What are the horizontal and vertical components of the total force moving the plow if the tension in each of the four traces is 60 pounds?

6. Why does one lower the handle of a lawn mower through tall grass?

7. A steamer is moving N 62° $20'$ E at the rate of 20 miles an hour. Find its rates of travel north and east.

8. A canal boat is 10 feet from the shore, and a pull of 200 lb. is exerted on the 60-foot towline. What is the effective component?

9. Why is a long rope more effective in hauling a canal boat than a short rope?

10. One of the two components acting at right angles is 4 times the other. Their resultant is 40 pounds. Find the forces.

11. A boy weighing 100 lb. sits in a swing. A man pushes the boy to one side with a force of 25 pounds. What is the magnitude and direction of the pull exerted by the rope?

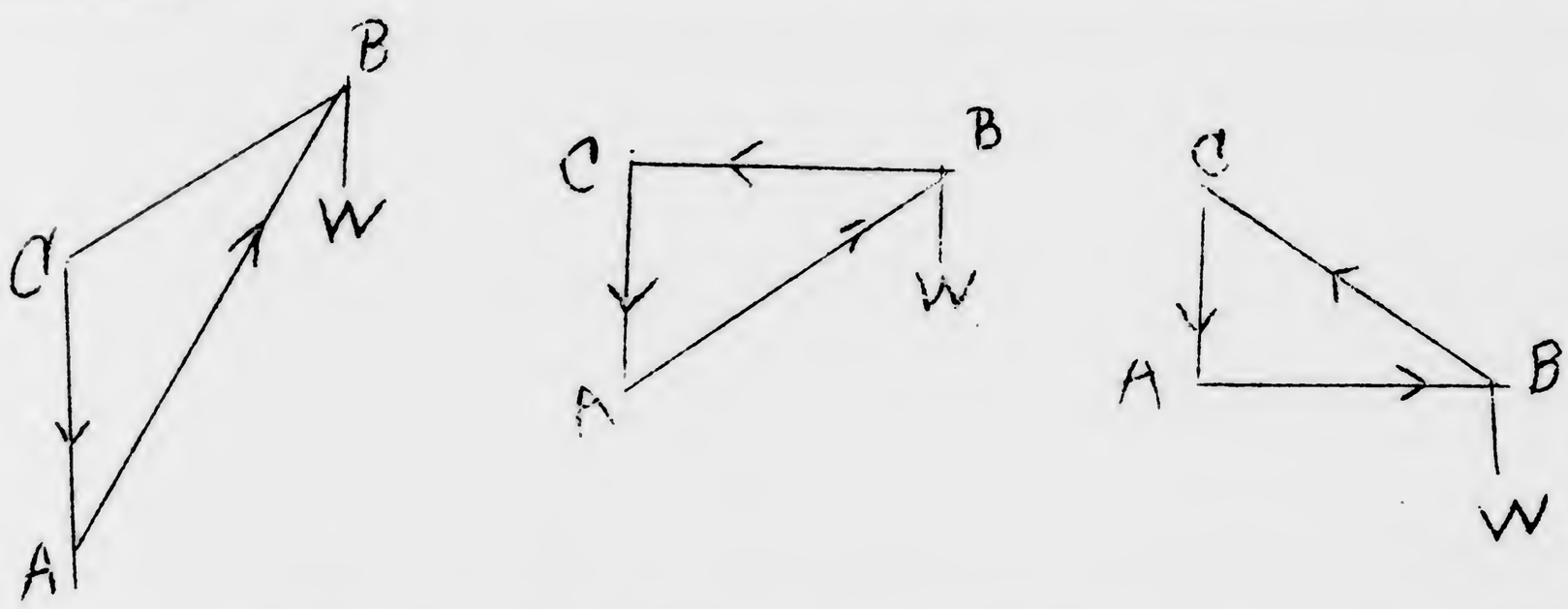
12. Show in general that the component of a force along any fixed line is equal to the magnitude of the force times the cosine of the angle between the direction of the force and the fixed line.

Equilibrant. The force necessary to balance or hold in equilibrium two or more given forces is called their equilibrant. The equilibrant of any number of forces is equal to their resultant but opposite in direction.

When three forces are in equilibrium, they may be represented in magnitude and direction by the three sides of a triangle, taken in order the same way round. The solution of this triangle helps to solve many problems in statics. (Statics is that branch of mechanics which treats of bodies at rest and of bodies whose motion is not changing.).

The Simple Crane.

The ordinanry jib-crane is a very useful application of the triangle of forces. It consists of a crane post A C, a jib A B, and a tie-rod B C. The triangle represents three forces, the weight, the thrust in the jib, and the pull in the tie-rod.



Example. A weight of 5 tons is lifted by a crane whose jib is 24 feet long and makes an angle of 60° with the horizontal. Find the thrust in the jib and the tension in the rod if the post is 15 feet long.

Since A C = 15 ft. represents 5 tons,

A B = 24 ft. represents $\frac{24}{15} \times 5 = 8$ tons of thrust.

To find B C, we have from the law of cosines

$$B\,C = \sqrt{24^2 + 15^2 - 2 \times 24 \times 15 \cos 30^\circ}$$

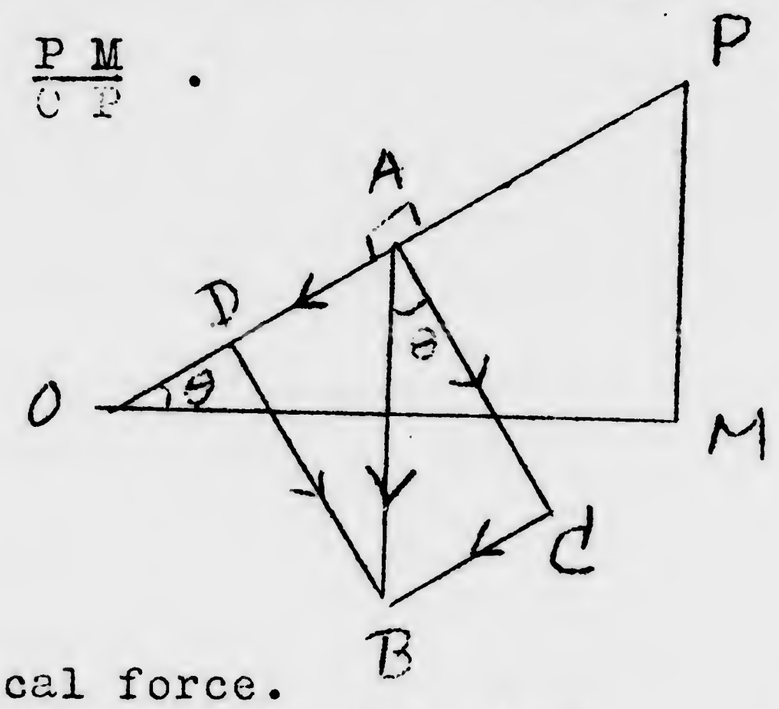
$$= \sqrt{177.48} = 3.3 \text{ tons, the pull in the tie-rod.}$$

FADED/LIGHT TEXT

The Inclined Plane.

Consider a body resting on a plane inclined at an angle θ to the horizontal. The weight of the body acts vertically downward. It gives rise to two effects, or it is resolved into two forces. The first force acts down the plane and is represented by A D; the second force is the pressure of the body on the plane and perpendicular to it, and is represented by A C. These two forces are the components of the weight of the body. We have to construct the parallelogram of forces to find their magnitudes (since their directions are given). Triangles O M P and B D A are similar. (Why?)

Hence, $\frac{A\ D}{A\ B} = \frac{P\ M}{O\ P}$.



AB = W, the vertical force.
AC = W cos θ, the force ⊥ to the plane.
AD = CB = W sin θ, the force ‖ to the plane.

Example.

A box weighing 200 lbs. is placed on a plane which is inclined at an angle of 28°. How great must friction be to keep the box from sliding, and what pressure does the box exert upon the plane?

Component down the plane = 200 sin 28° = 93.89 lb.

Pressure on the plane = 200 cos 28° = 176.59 lb.

Exercise.

(Draw a figure for each exercise)

1. Two forces of 100 lb. and 160 lb. act at a point and make an angle of 75° with each other. Find the magnitude and direction of their equilibrant.

2. A weight of 200 lb. is suspended by two strings which make angles of 60° with the horizontal. Find their tensions (pulls).

3. A weight of 75 lb. is supported by two strings making equal angles with the horizontal and an angle of 90° with each other. What is the tension on each string? What would be the tension on each string if one made an angle of 30° with the horizontal and the other 60°?

4. A barrel weighing 80 lb. is placed on a plane inclined at an angle of 40°. What force tends to cause motion down the plane? What is the pressure on the plane?

5. A pig weighing 150 lb. is supported by a smooth ring on a rope between two posts 10 feet apart. What is the tension in the rope when the sag is 6 ft? When the sag is one foot? When the sag is 3 inches

6. Why will a long rope hanging between two points at the same level break before it can be pulled tight enough to be straight?

7. An automobile weighing 2600 lb. stands on a road inclined at 11°. What force is necessary to overcome by the brakes to prevent it rolling down? (Neglect friction).

8. A weight of one ton is suspended at the end of a jib 30 feet long by a rope 18 feet long. Find the thrust in the jib and the pull in the rope if the rope is attached to the crane post (a) 16 feet from its lower end, (b) 24 ft from its lower end.

9. A lamp weighing 40 lb. is hung out from a pole by means of a stiff rod 10 ft. long and perpendicular to the pole, and a tie wire fastened to the pole 4 feet above the foot of the rod. What is the pull of the wire and the thrust of the rod?

10. A picture weighing 30 lb. is hung by a cord passing over a nail, the two parts of the cord making an angle of 70° with each other. What is the tension of the cord?

RADIAN MEASURE

Measurement of Angles.

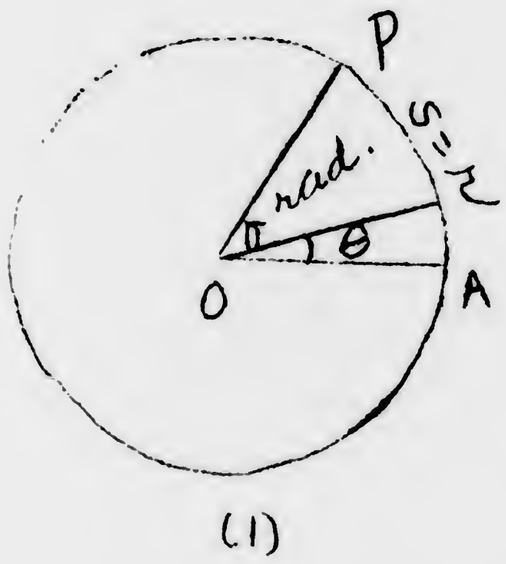
An angle is measured by finding its ratio to a given unit of measure. Various units have been used.

1. The Right Angle. This unit was used by the Greeks and is familiar to the student from elementary geometry. The scale of the mariner's compass is a practical application.

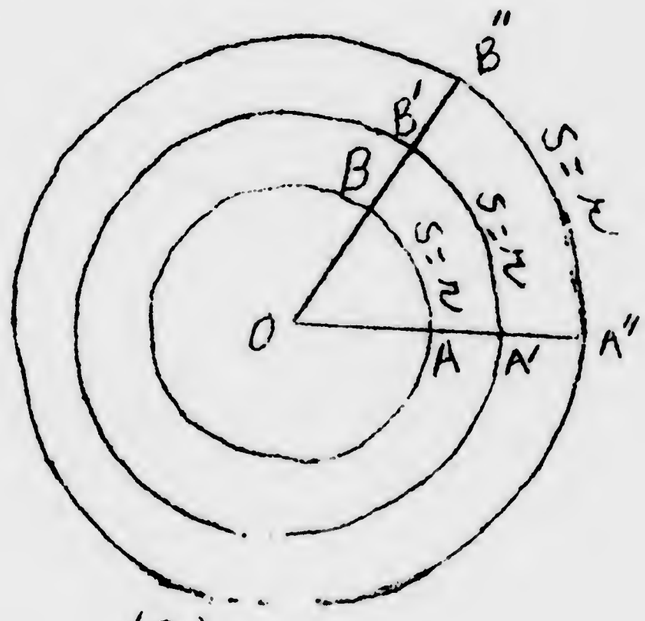
2. The Degree. The degree is $\frac{1}{360}$ of a rotation. It is an artificial unit originated by the Babylonians as early as 1000 B. C. It is the unit most generally used in practical problems involving numerical computations.

3. The Radian. A radian is an angle which at the center of a circle intercepts an arc equal in length to the radius of the circle. The radian was introduced early in the nineteenth century. It is of fundamental importance in physics and higher mathematics. It is often called the circular measure.

We know from geometry that the ratio of the circumference to the radius is the same for all circles. From this it follows that the radian has a fixed value, or is a constant angle, and does not depend upon the size of the circle.



(1)



(2)

Relation Between Degree and Radian.

Since the radius is contained 2π times in a circumference, there are 2π radians in a complete rotation. Hence

(1)
$$2\pi \text{ radians} = 360^\circ$$
$$\pi \text{ radians} = 180^\circ$$
$$\frac{\pi}{2} \text{ radians} = 90^\circ, \quad \frac{\pi}{6} \text{ radians} = 30^\circ.$$
$$1 \text{ radian} = \frac{180^\circ}{\pi} = \frac{180^\circ}{3.1416} = 57.296^\circ \text{ (or } 57.3^\circ)$$
$$1 \text{ radian} = 57^\circ\ 17'\ 45''$$
$$1^\circ = \frac{\pi}{180} \text{ radian} = 0.01745 \text{ radian}$$

Formula (1), π radians $= 180^\circ$, vonnects the two units and is easy to remember.

To convert radians to degrees, multiply the number of radians by $\frac{180}{\pi}$ or 57.3.

To convert degrees to radians, multiply the number of degrees by $\frac{\pi}{180}$, or 0.01745.

Example. 1. Express $8^\circ\ 15'$ in radians.

$180^\circ = \pi r$, $1^\circ = \frac{\pi}{180}$

$$8^\circ\ 15' = \frac{33^\circ}{4} = \frac{33}{4} \times \frac{\pi}{180} = 0.144 \text{ radian.}$$

Example 2. Express 2.3 radians in degrees.

π rad. $= 180^\circ$, 1 rad. $= \frac{180}{\pi}$

$$2.3 = 2.3 \times \frac{180}{\pi} = 131.79^\circ$$

In writing an angle in radian measure, usually no symbol for radian is used. When no angular unit is expressed, the radian is always understood to be the unit. The angle 3 means 3 radians, the angle $\frac{\pi}{2}$ means $\frac{\pi}{2}$ radians.

The student should remember that π is a numerical quantity equal to 3.1416 and can never equal 180°. The expression $180^\circ = \pi$ does not mean $180^\circ = 3.1416$; it means $180^\circ = 3.1416$ radians. The most frequently used angles are conveniently expressed in radian measure by using π. Thus,

$$60^\circ = \frac{\pi}{3}, \quad 45^\circ = \frac{\pi}{4}, \quad 135^\circ = \frac{3\pi}{4}, \text{ etc.}$$

Relation Between Angle, Radius and Length of Arc.

From the definition of a radian it is clear that the number of radians in an angle is the number of times the intercepted arc contains the radius.

That is,

$$\frac{\text{length of intercepted arc}}{\text{length of radius}} = \text{number of radians in an angle.}$$

Hence,

arc = radius × radians.

(2) or $s = r\theta$

where s = length of arc, r = length of radius, and θ = the radian measure of the angle. Formula (2) contains three variables. If any two of them are known, the third can be found.

Example. If the radius of a circle is 10 feet, what is the length of arc intercepted by a central angle of 120°?

$$120^\circ = \frac{2\pi}{3} \text{ radians}$$

$$s = r\theta = 10 \times \frac{2\pi}{3} = \frac{20\pi}{3} = 20.944 \text{ feet.}$$

Linear and Angular Velocity.

If a point P moves along the circumference of a circle (fig. 1), the radius O P rotates and generates the angle θ. The rate v at which P is moving is called the linear velocity. The rate w at which θ is moving is called the angular velocity. For every radian through which O P turns the point P travels a distance equal to the radius. Formula (2) $s = r\theta$ connects the two velocities and is written in the special form, $v = rw$.

Linear velocity = radius × angular velocity.

The angular velocity of rotating parts is usually given either in revolutions per minute (r.p.m.) or in radians per second.

Exercise.

1. Express the following angles in degrees.
   2 rad., $\frac{1}{2}$ rad., 3.4 rad., $\frac{2}{7}$ rad., $-\frac{5}{3}$ rad., $1\frac{3}{8}$ rad.

2. Express in degrees: $\frac{\pi}{7}$ rad., $-\frac{\pi}{3}$, $\frac{3\pi}{2}$, $\frac{7\pi}{3}$, $\frac{2\pi}{5}$.

3. Express in radians: 80°, 125°, 810°, $-12^\circ 15'$, $75^\circ 35'$.

4. Express in radians: 30°, 45°, 60°, 90°, 120°, 135°, 150°, 180°, 210°, 225°, 270°, 300°.

5. What is the relation between a right angle and a radian? How many right anglesa are there in each of the following angles? $\frac{2\pi}{7}$, $\frac{3}{5}\pi$, 4 rad., 6.5 rad., 115°, 42°.

6. Through how many radians do the minute and hour hands of a clock turn in (a) 30 min.? (b) 10 min.? (c) 16 min.? (d) 1 min.? (e) 1 sec.?

7. If an arc of 24 ft. subtends an angle of 3 radians, find the radius of the circle.

8. If an arc of 14.3 ft. subtends an angle of 72°, find the radius of the circle.

9. The length of the arc of a colored sector of a circle is 2.3 ft., the radius is 1.5 ft. Find the angle of the sector. Find the area of the sector.
(Note. Area of sector = $\frac{1}{2}$ r.s = $\frac{1}{2}$ $r^2\theta$).

10. A pendulum 39 inches long vibrates $3^\circ\ 10'$ on each side of its vertical position. Find the length of arc through which it swings.

11. Find the distance measured on the surface of the earth from the equator to a city 35° north latitude to State College, Pennsylvania. (r = 3960 m.)

12. A wheel on a dynamo turns 80 r.p.m. The wheel has a diameter of 12 ft. Find the linear velocity of a point on the rim of the wheel. How many radians does it turn in one second?

13. The front wheel of a cart is 2.5 ft. in diameter, the hind wheel 3.5 ft. How many revolutions does each wheel make if the cart travels one mile? Through how many radians will the hind wheel turn if the front wheel turns through 800°?

14. The diameter of a graduated circle is 4 feet, and the graduations on the circumference are 5 minutes apart. Find the length of arc between successive graduations.

15. The diameter of the sun subtends an angle of $32'\ 2''$ at the earth, and the distance of the sun is 92,750,000 miles. Find the diameter of the sun.
(Note. When distances are very great and angles are very small, the subtended arc is approximately equal to the diameter.)

16. If the earth subtends an angle $8.8''$ at the sun(sun's parallax), find the distance of the sun from the earth.

Functions of Two Angles.

In the preceding sections we considered the properties and relations of the functions of a single angle. We solved equations and verified identities involving the functions of one angle only. But, in higher mathematics and in many of their applications, we often meet relations involving the functions of two or more angles. In this section we shall develop some of these formulas which are interesting in themselves, and which are needed for the study of advanced branches of mathematics.

Many of the proofs and exercises of this section are useful and important, but the student should direct his attention mainly to the methods whereby the results are obtained. This will help to develop in him a mathematical type of reasoning and give him power to proceed quickly and confidently to solve similar problems.

The work of this section requires a ready and accurate knowledge of the formulas developed in the preceding sections. The student should review those formulas and learn them carefully.

The Sine of the Sum of Two Angles.

In figure (1) let A B and (A+B) be acute angles. From any point P on the terminal side of (A+B), drop two perpendiculars, P C to the initial side, and P D to the terminal side of angle A. Then from D drop two perpendiculars, D E to the initial side of angle A, and D F to P C. The construction makes angle F P D equal to angle A. (Why?)

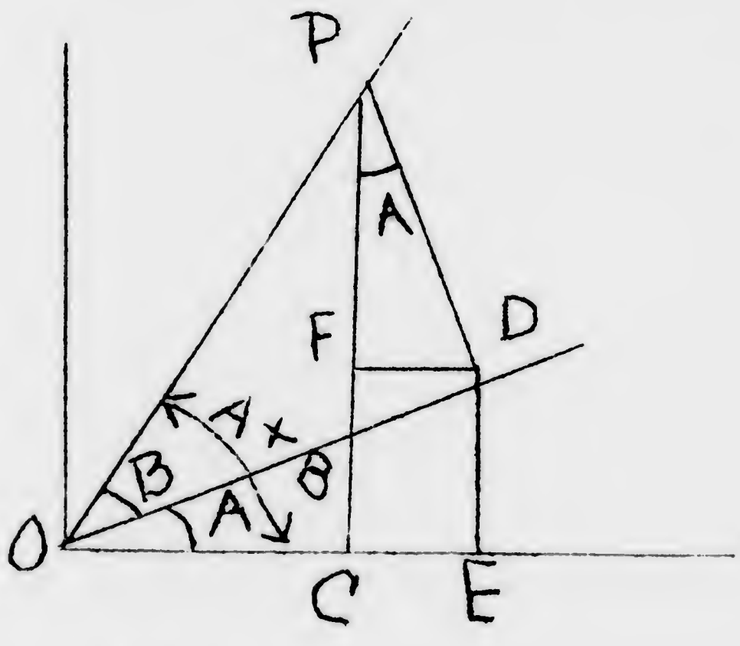


Fig. (1)

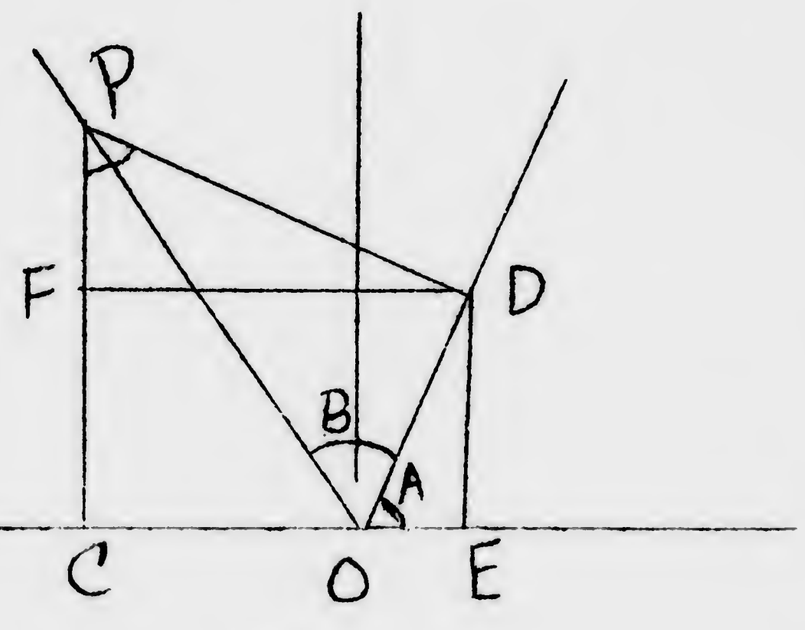


Fig. (2)

By definition,

$$\sin(A + B) = \frac{CP}{OP} = \frac{ED + FP}{OP} = \frac{ED}{OP} + \frac{FP}{OP}$$

$$= \frac{ED}{OP} \cdot \frac{OD}{OD} + \frac{FP}{CP} \cdot \frac{DP}{DP}$$

$$= \frac{ED}{OD} \cdot \frac{OD}{OP} + \frac{FP}{DP} \cdot \frac{DP}{OP}$$

(1) $\therefore \quad \sin(A+B) = \sin A \cos B + \cos A \sin B$

Note. The fraction $\frac{ED}{OP}$ is a ratio of sides of two different right triangles which have a common side OD. We multiplied $\frac{ED}{OP}$ by $\frac{OD}{OD}$ and arranged the product so as to get two ratios each of which contains two sides of the same right triangle. In the same way we multiplied $\frac{FP}{OP}$ by $\frac{DP}{DP}$.

In the proof we assumed that A, B, and (A+B) were acute angles. It can be shown, however, that the formula is true for all angles. It may be stated in words as follows:

The sine of the sum of any two angles equals the sine of the first angle times the cosine of the second, plus the cosine of the first times the sine of the second.

Example. $\sin(30^\circ + 45^\circ) = \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ$.

Thus $\sin 75^\circ$ can be calculated from the functions of 30° and 45° which we already know from simple right triangles.

The Cosine of the Sum of Two Angles.

From figure (1) above we have by definition:

$$\cos(A+B) = \frac{OC}{OP} = \frac{OE - CE}{OP} = \frac{OE}{OP} - \frac{CE}{OP}$$

$$= \frac{OE}{OP} - \frac{FD}{OP} = \frac{OE}{OP} \cdot \frac{OD}{OD} - \frac{CE}{OP} \cdot \frac{DP}{DP}$$

$$= \frac{OE}{OD} \cdot \frac{OD}{OP} - \frac{CE}{DP} \cdot \frac{DP}{OP}$$

(2) $\therefore$ $\cos(A+B) = \cos A \cos B - \sin A \sin B$.

This formula is valid for all angles, and it may be stated in words as follows:

> The cosine of the sum of any two angles equals the product of their cosines minus the product of their sines.

The Tangent of the Sum of Two Angles.

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

Dividing each term of the numerator and denominator by $\cos A \cos B$, we obtain

$$\tan(A+B) = \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A}{\cos A} \cdot \frac{\sin B}{\cos B}}$$

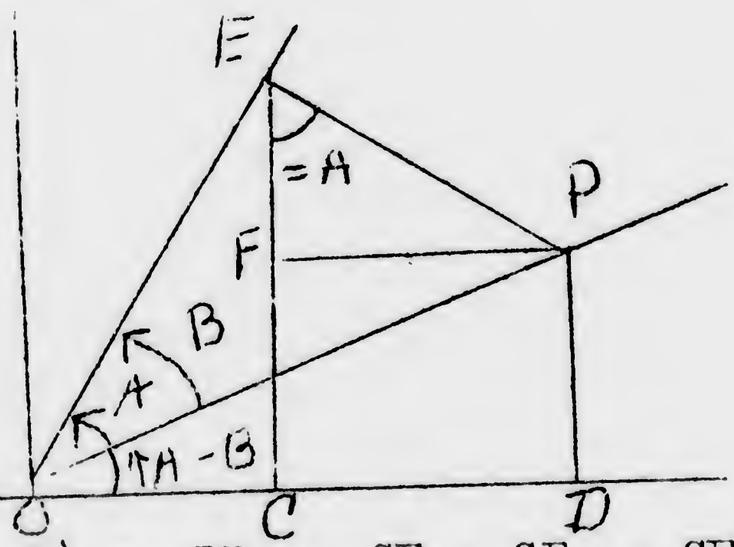
(3) That is, $$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

The student should state this formula in words.

The Sine of the Difference of Two Angles.

Place the acute angles A and B in the standard position as shown in the figure. Then angle DOP = (A - B). Choose any point P on the terminal side of (A - B). From P draw PD perpendicular to OD and PE perpendicular to OE. From E draw EC perpendicular to OD, and from P draw PF perpendicular to EC. Then angle FEP = A (Why?)

Fig (3)



By definition,

$$\sin(A - B) = \frac{DP}{OP} = \frac{CE - CF}{OP} = \frac{CE}{OP} - \frac{FE}{OP}$$

$$= \frac{CE}{OE} \cdot \frac{OE}{OP} - \frac{FE}{EP} \cdot \frac{EP}{OP}$$

(4) $\therefore$ $\sin(A - B) = \sin A \cos B - \cos A \sin B$

Example. $\sin(60^\circ - 45^\circ) = \sin 60^\circ \cos 45^\circ - \cos 60^\circ \sin 45^\circ$.
This helps us to calculate the sine of 15°.

The Cosine of the Difference of Two Angles.

From figure (3) we have by definition,

$$\cos(A - B) = \frac{OD}{OP} = \frac{OC + CD}{OP} = \frac{OC}{OP} + \frac{FP}{OP}$$

$$= \frac{OC}{OE} \cdot \frac{OE}{OP} + \frac{FP}{EP} \cdot \frac{EP}{OP}$$

(5) $\therefore$ $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

The Tangent of the Difference of Two Angles.

$$\tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)} = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$$

Dividing the numerator and denominator by $\cos A \cos B$ and simplifying, we obtain

(6) $$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

Formulas (4), (5) and (6) are entirely general. The student should state them in words. These three formulas can be easily derived from (1), (2) and (3). Thus

$$\sin(A - B) = \sin\left[A + (-B)\right] = \sin A \cos(-B) + \cos A \sin(-B).$$
$$= \sin A \cos B - \cos A \sin B.$$

since $\cos(-B) = \cos B$, and $\sin(-A) = -\sin A$.

In a similar way the student should develop $\cos(A - B)$ and $\tan(A - B)$.

Summary of Addition and Subtraction Formulas.

Collecting the above formulas for convenience of reference, we have

(1) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(2) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

(3) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(4) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

(5) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(6) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Observe that the subtraction formulas are just like the addition formulas except for the signs.

Exercise.

1. From the functions of 60° and 45° find the sine, cosine and tangent of 105°.

2. From the functions of 45° and 30°, find the sine, cosine and tangent of 15°.

3. Using the addition and subtraction formulas, find the sines, cosines and tangents of

(a) 75° (b) 120° (c) 210°

(d) 135° (e) 240° (f) 315°

4. Expand and simplify

(a) $\sin(90^\circ + A)$ (d) $\sin(180^\circ - A)$

(b) $\cos(90^\circ + A)$ (e) $\cos(180^\circ - A)$

(c) $\tan(90^\circ + A)$ (f) $\tan(180^\circ - A)$

5. Prove formulas (1) and (2) from figure (2).

6. Find formulas for $\cot(A+B)$ and $\cot(A - B)$ in terms of the cotangents of A and B.

7. (a) Is $\sin(90^\circ - 60^\circ) = \sin 90^\circ - \sin 60^\circ$?

(b) Is $\cos(30^\circ + 60^\circ) = \cos 30^\circ + \cos 60^\circ$?

(c) Is $\tan(x+y) = \tan x + \tan y$?

This is the case of the "trigonometric asinorum".

8. Expand and simplify.
(a) $\sin(\theta + 45^\circ)$. (b) $\cos(3t - 60^\circ)$.

(c) $\tan(45^\circ + A) + \tan(45^\circ - A)$.

9. Given $\sin A = \frac{3}{5}$, $\cos B = \frac{1}{2}$, A and B less than 90°, find $\sin(A+B)$, $\cos(A - B)$, $\tan(A+B)$.

10. Given $\sin A = \frac{1}{2}$, $\cos B = \frac{\sqrt{3}}{2}$, A and B being acute angles, find $\sin(A - B)$, $\cos(A+B)$, $\tan(A - B)$.

11. If $\cos A = .52$ and $\sin B = .80$, A and B in the first quadrant, find $\sin(A - B)$, $\cos(A - B)$, $\tan(A - B)$.

12. If $\sin x = .5$ and $\tan y = \frac{1}{\sqrt{3}}$, find $\sin(x+y)$, $\cos(x+y)$, $\tan(x+y)$. Find also $(x+y)$.

13. $\sin A = -\frac{1}{3}$, $\cos B = \frac{1}{4}$, A is in the third quadrant and B in the fourth, find $\sin(A+B)$, $\cos(A - B)$, $\tan(A+B)$.

14. If $\tan A = \frac{2}{3}$ and $\tan B = \frac{3}{2}$. Find $(A+B)$.

15. Expand and simplify.

(a) $\sin(A+B) + \sin(A - B)$
(b) $\sin(A+B) - \sin(A - B)$
(c) $\cos(A+B) + \cos(A - B)$
(d) $\cos(A+B) - \cos(A - B)$

16. Derive the formula for $\sin(A + B + C)$ in terms of the functions of A, B, and C.

17. Given $\sin A = \frac{3}{4}$ and $\sin B = \frac{5}{13}$, find $\sin(A - B)$, $\cos(A - B)$, $\tan(A - B)$ when A and B are acute; when A and B are both obtuse. Explain the results. Find $\sin(A - B)$, $\cos(A - B)$, $\tan(A - B)$ when A is obtuse B is acute. Interpret.

18. Find the value of

(a) $\sin(\tan^{-1} u)$

(b) $\sin(\sin^{-1} \frac{1}{2} + \cos^{-1} \frac{\sqrt{3}}{2})$

(c) $\cos(\sin^{-1} \frac{1}{3} + \cos^{-1} \frac{1}{2})$

(d) $\tan(\tan^{-1} \frac{2}{5} - \tan^{-1} \frac{1}{5})$

Functions of Double Angles.

To find the functions of an angle in terms of functions of an angle half as large.

From formula (1) letting B = A, we have,

$\sin(A+A) = \sin A \cos A + \cos A \sin A.$

(7) $\therefore \sin 2A = 2 \sin A \cos A.$

In words, the sine of any angle equals two times the sine of half the angle times the cosine of half the angle.

Similarly, from (2) $\cos(A+A) = \cos A \cos A - \sin A \sin A.$

(8) $\therefore \cos 2A = \cos^2 A - \sin^2 A.$

$= 1 - 2 \sin^2 A$ (Why?)

$= 2 \cos^2 A - 1$

(9) $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$

The student should translate formulas (8) and (9) into words.

Illustrations:

$\sin 60^\circ = 2 \sin 30^\circ \cos 30^\circ$

$\sin 10A = 2 \sin 5A \cos 5A$

$\cos A = \cos^2 \frac{1}{2} A - \sin^2 \frac{1}{2} A$

$\tan 6A = \frac{2 \tan 3A}{1 - \tan^2 3A}$

Functions of Half Angles.

To find the functions of an angle in terms of functions of an angle twice as large.

From (8) we have

$$\cos 2A = 1 - 2\sin^2 A$$

$$2\sin^2 A = 1 - \cos 2A$$

$$\sin^2 A = \frac{1 - \cos 2A}{2}$$

(10) $\therefore$ $$\sin A = \sqrt{\frac{1 - \cos 2A}{2}}$$

or $$\sin \frac{A}{2} = \sqrt{\frac{1 - \cos A}{2}}$$

Also from (8), $$\cos 2A = 2\cos^2 A - 1$$

$$\cos^2 A = \frac{1 + \cos 2A}{2}$$

(11) $\therefore$ $$\cos A = \sqrt{\frac{1 + \cos 2A}{2}}$$

or $$\cos \frac{A}{2} = \sqrt{\frac{1 + \cos A}{2}}$$

From (10) and (11), by division, we get

(12) $$\tan A = \sqrt{\frac{1 - \cos 2A}{1 + \cos 2A}}$$

or $$\tan \frac{A}{2} = \sqrt{\frac{1 - \cos A}{1 + \cos A}}$$

Formulas (10), (11), and (12) are called the cosine formulas to help the memory. The student should state them in words thus:

> The sine of any angle is equal to the square root of a fraction whose numerator is one minus the cosine of an angle twice as large, and whose denominator is two.

Illustrations:

1. $\sin 30^\circ = \sqrt{\frac{1 - \cos 60^\circ}{2}} = \sqrt{\frac{1 - \frac{1}{2}}{2}} = \frac{1}{2}$.

2. $\cos 22\frac{1}{2}^\circ = \sqrt{\frac{1 + \cos 45^\circ}{2}}$

3. $\tan 4A = \sqrt{\frac{1 - \cos 8A}{1 + \cos 8A}}$

Formulas (1) to (12) enable us to construct a table of the values of the trigonometric functions. This was the method used before the discovery of the calculus.

Exercise.

1. From the functions of 30°, find the sine, cosine, and tangent of 60°.

2. From the functions of 60°, find the sine, cosine, and tangent of 120°.

3. From the cosine of 30°, find the sine, cosine and tangent of 15°, $7\frac{1}{2}^\circ$.

4. From the cosine of 45°, find thefunctions of $22\frac{1}{2}^\circ$, $11\frac{1}{4}^\circ$.

5. Describe how tables of the values of trigonometric functions may be computed.

6. Given $\sin A = .6$, and A acute, find:

   (a) the sine, cosine and tangent of 2 A.

   (b) the sine, cosine and tangent of $\frac{A}{2}$

7. Given $\cos A = \frac{3}{5}$, and A acute, find

   (a) the functions of 2 A

   (b) the functions of $\frac{A}{2}$

8. Given $\tan \theta = \frac{4}{3}$, find

   (a) the functions of 2θ

   (b) the functions of $\frac{\theta}{2}$

9. Given $\cos \theta = -\frac{1}{4}$, and θ obtuse, find

   (a) the functions of 2θ

   (b) the functions of $\frac{\theta}{2}$

10. Find the value of

(a) $\sin(\frac{1}{2}\cos^{-1}\frac{2}{3})$. (Let $\theta = \cos^{-1}\frac{2}{3}$)

(b) $\tan(2\sin^{-1}\frac{1}{4})$

(c) $\cos(\frac{1}{2}\tan^{-1}3)$

Products of Sines and Cosines Expressed as Sums, and Sums Expressed as Products.

From the addition formulas, we have

(1) $\sin(A+B) = \sin A\cos B + \cos A\sin B.$
(4) $\sin(A-B) = \sin A\cos B - \cos A\sin B.$
(2) $\cos(A+B) = \cos A\cos B - \sin A\sin B.$
(5) $\cos(A-B) = \cos A\cos B + \sin A\sin B.$

From (1) and (4) by addition and subtraction, we get

(13) $\sin(A+B)+\sin(A-B) = 2\sin A\cos B$

or $\sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$

(14) $\sin(A+B) - \sin(A-B) = 2\cos A\sin B$

From (2) and (5), by addition and subtraction, we get

(15) $\cos(A+B)+\cos(A-B) = 2\cos A\cos B$

(16) $\cos(A+B) - \cos(A-B) = -2\sin A\sin B.$

Formulas (13) to (16) enable us to transform a product of sines and cosines into a sum or a difference. The principal use of these formulas is in certain transformations in the integral calculus. Before the invention of logarithms they were used for purposes of calculation because addition and subtraction were found simpler than multiplication. At present, however, we prefer to compute from products by the use of logarithms. By a slight modification, formulas (13) to (16) give us formulas for converting sums and differences into products.

Let $A+B = X$

and $A-B = Y$

Then $2A = X+Y$, and $2B = X - Y$

$A = \frac{1}{2}(X+Y)$, and $B = \frac{1}{2}(X - Y)$

Substituting the values of A and B in formulas (13) to (16), we have

(17) $\sin X+\sin Y = 2\sin\frac{1}{2}(X+Y)\cos\frac{1}{2}(X - Y)$

(18) $\sin X - \sin Y = 2 \cos\frac{1}{2}(X + Y) \sin\frac{1}{2}(X - Y)$

(19) $\cos X + \cos Y = 2 \cos\frac{1}{2}(X + Y) \cos\frac{1}{2}(X - Y)$

(20) $\cos X - \cos Y = -2 \sin\frac{1}{2}(X + Y) \sin\frac{1}{2}(X - Y)$

Illustrations:

1. $\sin 50^\circ \cos 20^\circ = \frac{1}{2}\left[\sin(50^\circ + 20^\circ) + \sin(50^\circ - 20^\circ)\right]$
$= \frac{1}{2}(\sin 70^\circ + \sin 30^\circ)$.

2. $\cos 60^\circ + \cos 30^\circ = 2 \cos \frac{1}{2}(60^\circ + 30^\circ) \cos\frac{1}{2}(60^\circ - 30^\circ)$
$= 2 \cos 45^\circ \cos 15^\circ$.

3. $\cos 7A \cdot \cos 3A = \frac{1}{2}(\cos 10A + \cos 4A)$

4. $\sin 12A - \sin 10A = 2 \cos 11A \sin A$.

Exercise.

1. State formulas (13) to (20) in words.
Find the values of the functions from the tables, and verify the following equations:

2. $\cos 40^\circ \sin 20^\circ = \frac{1}{2}(\sin 60^\circ - \sin 20^\circ)$.

3. $\cos 70^\circ \cos 40^\circ = \frac{1}{2}(\cos 110^\circ + \cos 30^\circ)$.

4. $\sin 50^\circ + \sin 10^\circ = 2 \sin 30^\circ \cos 20^\circ$.

5. $\cos 40^\circ - \cos 26^\circ = -2 \sin 33^\circ \sin 7^\circ$.

Express the following products as sums and differences:

6. $\sin 8A \cos 2A$.

7. $\sin 5A \sin 3A$.

8. $\cos 12A \sin 4A$.

9. $\cos 7A \cos 9A$.

10. $\sin 54^\circ \cos 32^\circ$.

11. $\cos 80^\circ \sin 30^\circ$.

12. $\cos(X - Y) \cos Y$.

13. $\sin(45^\circ + A) \cos(45^\circ - A)$

Express the following as products:

14. $\cos 50^\circ + \sin 30^\circ$.

15. $\sin 45^\circ + \sin 25^\circ$.

16. $\sin 24^\circ - \sin 36^\circ$.

17. $\cos 7A + \cos 3A$.

18. $\dfrac{\cos 10^\circ - \cos 4^\circ}{\cos 10^\circ + \cos 4^\circ}$.

19. $\dfrac{\sin 70^\circ - \sin 40^\circ}{\sin 70^\circ + \sin 40^\circ}$.

20. $\frac{\cos 100^\circ + \cos 50^\circ}{\sin 100^\circ - \sin 50^\circ}$

21. $\frac{\sin 8A + \sin 6A}{\cos 8A - \cos 6A}$.

22. $\frac{\cos 5A - \cos 3A}{\sin 5A + \sin 3A}$

23. $\frac{\sin 9A + \sin 5A}{\cos 9A + \cos 5A}$

The Law of Tangents.

The law of tangents, which is used for the solution of triangles when two sides and the included angle are given, has been developed by means of a geometrical figure. It can be easily derived by combining the law of sines and the sums and products formulas as follows.

We have by the law of sines,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = D.$$

Hence, $a = D \sin A$, $b = D \sin B$, $c = D \sin C$.

and
$$\frac{a - b}{a + b} = \frac{\sin A - \sin B}{\sin A + \sin B}$$

$$= \frac{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A - B)}{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A - B)}$$

$$\therefore \quad \frac{a - b}{a + b} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A+B)}$$

The Tangent of Half the Angle.

The formula $\tan \frac{1}{2}A = \frac{r}{s - a}$ has been derived by using the law of cosines and a geometrical figure. It can be derived also by combining the law of cosines and the formulas of this section as follows:

From the law of cosines, $\cos A = (b^2 + c^2 - a^2) \div 2bc$

From formula (12), $\tan^2 \frac{1}{2}A = \frac{1 - \cos A}{1 + \cos A}$

Substituting the first into the second, and simplifying,

$$\tan^2 \tfrac{1}{2} A = \frac{2bc - (b^2 + c^2 - a^2)}{2bc + (b^2 + c^2 - a^2)} = \frac{a^2 - (b - c)^2}{(b+c)^2 - a^2}.$$

$$= \frac{(a + b - c)(a - b + c)}{(b + c + a)(b + c - a)} = \frac{2(s - a)\ 2(s - b)}{2s \cdot 2(s - a)}$$

$$\therefore \quad \tan\tfrac{1}{2}A = \sqrt{\frac{(s-a)(s-b)}{s(s-a)}}$$

$$\text{or} \quad \tan\tfrac{1}{2}A = \frac{r}{s-a} \cdot$$

$$(2s = (a+b+c), \text{ and } \quad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

In a similar way the student should derive formulas for $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$.

Identities.

The formulas developed in this section are identities involving the functions of two or more angles. They enable us to transform trigonometric expressions to more convenient forms. They are also useful in solving equations involving functions of multiple angles.

The work of verifying an identity is essentially the same as we have seen before. The student should first transform the functions of multiple angles into functions of the same angle. The resulting identity may then be verified as before. In connection with this section the student should review the fundamental relations and the identities studied in a previous section.

Exercise.

1. $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A.$

2. $\cos(A+B)\cos(A-B) = \cos 2A.$

3. $\sin(x+y)\cos y - \cos(x+y)\sin y = \sin x.$

4. $\cos^2 6x = 2\cos^2 3x - 1.$

5. $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot A - \cot B}$.

6. $\cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}$.

7. $\sin 3x = 3\sin x - 4\sin^3 x$.

8. $\cos 3x = 4\cos^3 x - 3\cos x$.

9. $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$.

10. $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$.

11. $\cos^4 A - \sin^4 A = \cos 2A.$

12. $\cos^4 A + \sin^4 A = 1 - \frac{1}{2}\sin^2 2A.$

13. $\sin(30^\circ + A) = \cos(60^\circ - A).$

14. $\sin(A + 60^\circ) + \sin(A - 60^\circ) = \sin A.$

15. $\cot A + \tan A = 2 \csc 2A.$

16. $\cot A - \tan A = 2 \cot 2A.$

17. $(\sin\frac{1}{2}A + \cos\frac{1}{2}A)^2 = 1 + \sin A.$

18. $\dfrac{\sec^2 A}{2 - \sec^2 A} = \sec 2A.$

19. $\dfrac{\cos A}{1 - \sin A} = \dfrac{1 + \tan \frac{1}{2} A}{1 - \tan \frac{1}{2} A}.$

20. $\tan 2A \tan A + 1 = \sec 2A.$

21. $\tan \frac{1}{2}A = \dfrac{1 - \cos A}{\sin A}$

22. $\tan \frac{1}{2} A = \dfrac{\sin A}{1 + \cos A}$

23. $(\sin \theta - \cos \theta)^2 = 1 - \sin 2\theta.$

24. $\tan(45^\circ + A) = \dfrac{1 + \tan A}{1 - \tan A}$

25. $\dfrac{1 + \sin A - \cos A}{1 + \sin A + \cos A} = \tan \frac{1}{2} A.$

26. $\dfrac{\sin 2x}{\sin x} - \dfrac{\cos 2x}{\cos x} = \sec x.$

27. $\tan A = \csc 2A - \cot A.$

28. $\tan^2 A + 2 \cot 2A \tan A = 1.$

29. $\tan^2 \frac{1}{2}A = 2 \csc A \tan \frac{1}{2}A + 1.$

30. $\tan \frac{1}{2}A = \dfrac{\cos 3A - \cos 4A}{\sin 3A + \sin 4A}$

31. $\sin 2A = \dfrac{2 \tan A}{1 + \tan^2 A}$

32. $\tan^2 \frac{1}{2} A = \dfrac{\tan A - \sin A}{\tan A + \sin A}$

33. $\tan A + \tan B + \tan C = \tan A \tan B \tan C.$ $(A + B + C = 180^\circ).$

34. $\sin^2 A \cos^2 A = \frac{1}{8}(1 - \cos 4A)$.

35. $\frac{\sin^2 A \cos^2 B - \cos^2 A \sin^2 B}{\cos^2 A \cos^2 B - \sin^2 A \sin^2 B} = \tan(A+B)\tan(A-B)$.

36. $\frac{\sin 3x + \sin x}{\sin 3x - \sin x} = \frac{2}{1 - \tan^2 x}$.

37. $\cos(\sin^{-1} A) = \sqrt{1 - A^2}$.

38. $\sin^{-1} \frac{a}{\sqrt{1 + a^2}} = \tan^{-1} a$.

39. $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1 - xy}$.

40. $\sin(\sin^{-1} u + \cos^{-1} v) = uv \pm \sqrt{1 - u^2}\sqrt{1 - v^2}$.

Equations.

We have learned to solve equations involving functions of a single angle. In the following exercise we have to solve equations involving functions of two or more angles. The student will find these suggestions helpful.

1. Reduce the given equations by factoring into simpler equations whenever you can.

2. By means of the fundamental identities change to functions of the same angle.

3. Change to the same function and solve as before.

Example. $\sin 5A + \sin A = \sin 3A$

$\therefore$ $2\sin 3A \cos 2A - \sin 3A = 0$

$\sin 3A(2 \cos 2A - 1) = 0$.

This reduces to two equations:

(a) $2 \cos 2A - 1 = 0$
$\cos 2A = \frac{1}{2}$
$2A = 60^\circ$ or 300°
$A = 30^\circ$ or 150°.

(b) $\sin 3A = 0$
$3A = 0^\circ$ or 180°.
$A = 0^\circ$ or 60°.

$\therefore$ The four values check in the original equation.

$\therefore$ $A = 0^\circ$, 30°, 60° or 150°.

The general value is $A = 2n\pi$, $2n\pi + 30^\circ$, $2n\pi + 60^\circ$, $2n\pi + 150^\circ$.

Exercise.

Solve the following equations:

1. $\sin 2x = 2\cos x$.
2. $\sec^2 A + \tan A = 7$.
3. $\cos 2A + 2\sin^2 \frac{1}{2}A = 1$.
4. $\cos 2x + \sin x = 1$.
5. $\sin 2A = \sin A$.
6. $\sin\theta \cos\theta = \frac{1}{2}$.
7. $4\sin A \cos 2A = 1$.
8. $\tan 2A \tan A = 1$.
9. $\sec A + \tan A = 2$.
10. $\sin\theta - \sin 3\theta = \cos 2\theta$.
11. $\cos 2x = \sin x$.
12. $\sin 2A = 1 + \cos 2A$.
13. $\cos x - \tan x = 0$.
14. $\cos 7A + \sin 4A = \cos A$.
15. $\sin 2x = \cos 3x$.
16. $\sin^2 2A - \sin^2 A = .25$.
17. $\sin 2x = \sin 3x$.
18. $\sin 7A = \sin 3A$.
19. $\sin 3x + \sin 2x + \sin x = 0$.
20. $\sin 2x + 3 = \sin x + 4\cos^2 \frac{1}{2}x$.
21. $\cos 3A + \cos A = \sin 2A$.
22. $\sin 5A - \sin 3A + \sin A = 0$.
23. $\cos 3y + \cos 2y + \cos y = 0$.
24. $2\sin A - 2\cos 2A + 1 = 0$.
25. $\sin^{-1} x + \sin^{-1} 2x = 60^0$.
26. $\tan^{-1}(m+1) + \tan^{-1}(m-1) = \tan^{-1} 2m$.

END OF YEAR